

ملخص مادة..

اهتزازات ميكانيكية

الهيئة
الميكانيكية
Polytechnic



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□

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

الحمد لله ربّ العالمين والصلاة والسلام على أشرف
الخلق والمرسلين سيدنا محمد وعلى آله وأصحابه أجمعين
أما بعد -

✳ المرجع في كتابة هذا التلخيص

Mechanical Vibrations

Fifth Edition in SI Units

كتاب ←

* ملاحظة * هذا التلخيص لا يُعني عن الكتاب المعتمد
وشرح مدرّس المادة

✳ طريقة الدراسة * مهم *

أولاً افهم الموضوع من التلخيص

ثانياً حل الأمثلة المختصة بهذا الموضوع من الكتاب

ثالثاً تمرّن على حل ال problems في آخر ال Chapter
فصوماً أسئلة الرسومات

رابعاً يمكن الرجوع للدرس الأخرى المحتوية على حل أسئلة - عندما تهمر -

* تنبيه

" لم أعتد في كتابة التلخيص على حل أية أمثلة

لكن اعتمدت كتابة شرح لأهم المفاهيم التي نحتاجها في هذا المساق

(دعائكم)



Translation Motion

Quantity	Units
1 mass " <u>m</u> "	kg
2 Stiffness, spring constant " <u>k</u> "	N/m
3 Viscous damping coefficient " <u>C</u> "	N.s/m
4 External Force " <u>F</u> "	N

$$\sum F = m \ddot{x}$$

Rotational Motion

Quantity	Units
1 mass moment of inertia " <u>J_o</u> "	kg.m ²
2 torsional spring constant " <u>k_t</u> "	N.m/rad
3 rotational damping constant " <u>C_t</u> "	N.m.s/rad
4 External Moment " <u>M</u> "	N.m

$$\sum M = J_o \ddot{\theta}$$

[*] When mass oscillates about a fixed point "O" or a pivot point "O"

Thus → The rotary inertia J_o is given by :-

$$J_o = J_G + md^2 \quad \text{where } d \text{ is distance from the center of gravity to the point "O"}$$

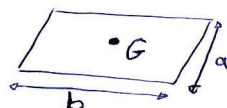
J_G : mass moment of inertia about Z-axis, normal to xy, and passing through the center of mass

Type
1 slender bar



$$J_G = \frac{1}{12} m l^2$$

2 Thin plate



$$J_G = \frac{1}{12} m [a^2 + b^2]$$

3 circular disk

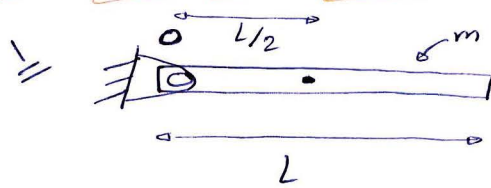


$$J_G = \frac{1}{2} m R^2$$

هذا ما يتعلمه
الفرقة من الإشارات
الرجاء أن تقرأ كتاب
Dynamic



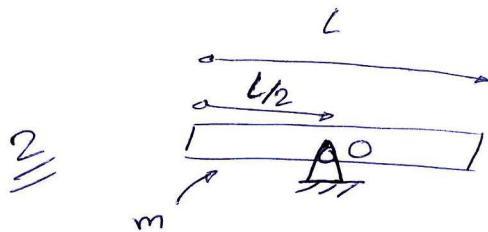
Examples to Find J_o



$$J_o = J_G + m d^2 \quad (d = \frac{L}{2})$$

$$J_o = \frac{1}{12} m L^2 + m \frac{L^2}{4}$$

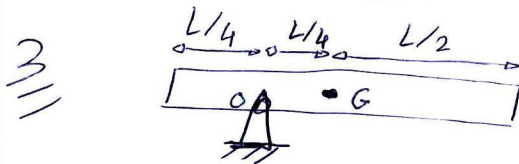
$$\rightarrow \boxed{J_o = \frac{m L^2}{3}}$$



$$J_o = J_G + m d^2 \quad (d = 0)$$

$$\boxed{J_o = \frac{m L^2}{12}}$$

لا تنس أن d هي المسافة من النقطة الثابتة ومركز الجسم "center of gravity"



$$J_o = J_G + m d^2$$

$$J_o = \frac{m L^2}{12} + m \frac{L^2}{16}$$

$$d = \frac{L}{4}$$

$$\boxed{J_o = \frac{7}{48} m L^2}$$

* The various classifications of vibration namely:-

- Free and Forced vibration
- Undamped and damped vibration
- Linear and nonlinear vibration
- Deterministic and random vibration

يتم شرحهم
في صفحة
Page 12 & 13

1.3 Importance of the study of vibration

- معظم ال Primemovers عتفا مشاكل اهتزاز

يسبب ← (unbalance in the engines) ← سبب في 3.1

- عدم الاتزان يمكن أن يكون بسبب عيب في التصميم أو سوء التصنيع

- عندما يتوافق التردد الطبيعي للاهتزاز آلة (machine) أو هيكل (structure)

مع تردد external (المثير الخارجي) excitation

تحدث ظاهرة الرنين Resonance $\omega = \omega_n$

(*) ملاحظة

الاهتزاز ليس سلبياً بالإطلاق !

فخلول اهتزاز لهبلاات البذن و موجلات الضوء والرئتين لما سحننا و رأينا و تنفسنا و غيرها من هوائد



1.4 Basic Concepts of Vibration

⊠ أي حركة تكرر نفسها بعد فترة زمنية ^{تسمى} Vibration or Oscillation

⊠ نظرية الاقتران تتعامل مع دراسة $\perp$ حركات الأجسام التذبذبية في القوى المرتبطة بها

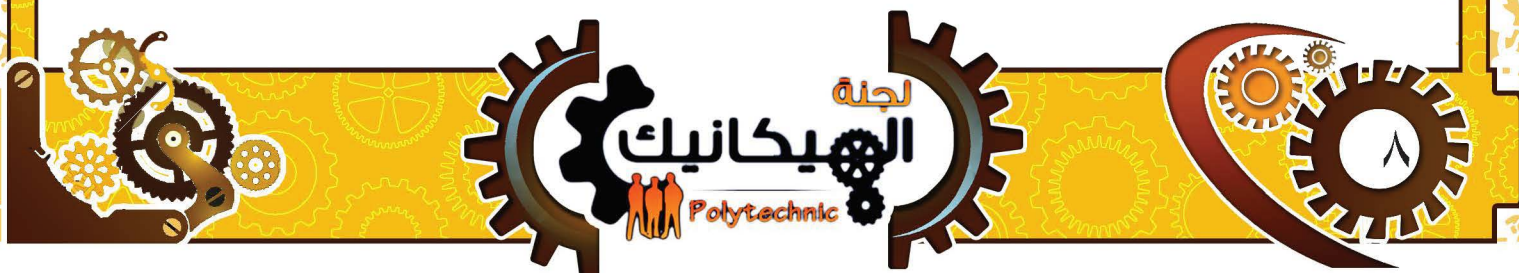
⊠ Elementary parts of vibrating systems

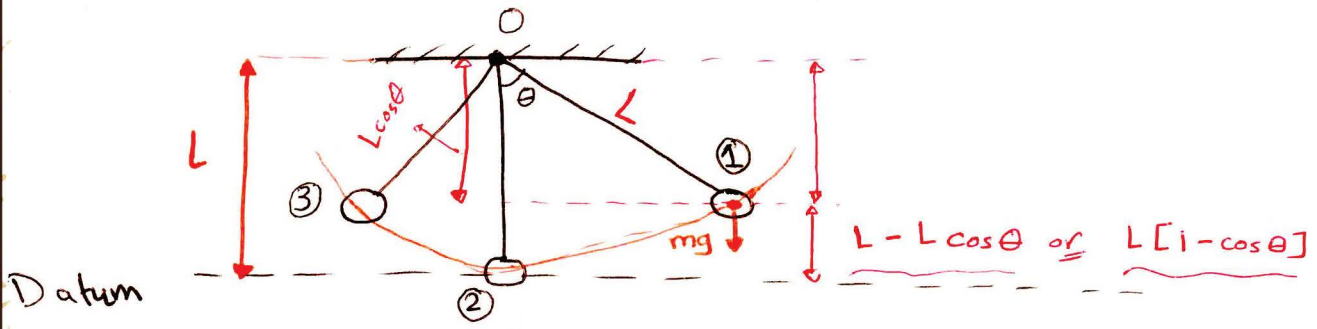
- A vibratory system, in general, includes:-

- 1] Spring/elasticity ∞ ∞ ∞ For storing potential energy
- 2] mass/inertia ∞ ∞ ∞ For storing Kinetic energy
- 3] Damper ∞ ∞ ∞ which energy is gradually Lost

⊠ The vibration of a system involves:-

→ The transfer of its potential energy $\rightarrow$ kinetic energy
والعكس and vice versa





$$K.E = \frac{1}{2} m v^2 \quad \text{Zero}$$

① إذا تركنا الكرة في هذا الوضع

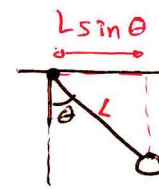
Zero السرعة ← Zero طاقة الحركة

$$P.E = mgh$$

$$= mg L [1 - \cos \theta]$$

← لكن طاقة الوضع

فكر لماذا تدفع الكرة من الموضع ① إلى ② مع أن طاقة الحركة = Zero



الجواب

$$T = F \cdot r \quad \text{Torque}$$

$$= mg L \sin \theta$$

بسبب ال Torque تبدأ الكرة بالتأرجح إلى اليسار

(مع عقارب الساعة) وبالتالي يكون هناك Angular Acceleration

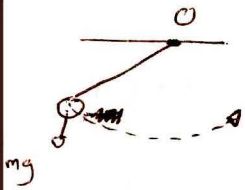
② كل طاقة الوضع تتحول إلى طاقة حركية
وتتحرّك حتى تصل الموقع ③

- عندما تمر الكرة من الموقع ② يبدأ مفعول ال Torque
بالجهة المعاكسة "اليمين" (عكس عقارب الساعة)
حتى تتساوى السرعة مع ال Torque فتتوقف الكرة عند ③

③ - لا توجد طاقة حركية

- طاقة الوضع $P.E = mgl[1 - \cos\theta]$

- ال Torque باتجاه عكس عقارب الساعة



✳ يستمر النظام على هذا الحال لفترة محدودة
وال θ "زاوية/مقدار التذبذب"

يقل في كل مرة تدريجياً

طاقة تنقل في كل دورة

(بسيط) - مقاومة الوسط المحيط "الهواء"

due to damping by the air



* Number of Degrees of Freedom "DoF"

تعريف → The min. number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time

one DOF

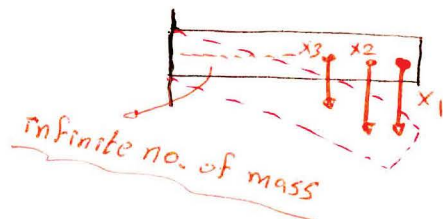
* سنحتاج فقط 8

وفي CH5 ستشعر ال Two DOF بشكل سطحي دون تعمق

وهو المطلوب منا في هذا ال course

* Discrete and Continuous Systems

1 Continuous or Distributed System



Infinite no. of mass

تعريف - System with infinite no. of DOF

Example :- Cantilever beam

2 Discrete or Lumped System

تعريف - system with finite no. of DOF

* في أغلب الأحيان يتم افتراض ال continuous بأنه Discrete

وعندما نحل باستخدام ال finite element ← عبارة عن مادة اختيارية في التحليل

* الموضوع ليس مهم في هذا ال course



1.5 Classification of Vibration

* Vibration can be classified in several ways

1] Free and Forced Vibration

Free vibration if a system is left to vibrate on its own,
"No external force"

ex3- A simple pendulum

Forced vibration if a system is subjected to ^{قوة خارجة} an external force

ex3- Diesel Engines

2] Undamped and Damped Vibration

Undamped vibration if no energy is lost or dissipated
in friction or other resistance
during oscillation

Damped Vibration if any energy is lost



3] Linear and Nonlinear Vibration

يسلك سلوك

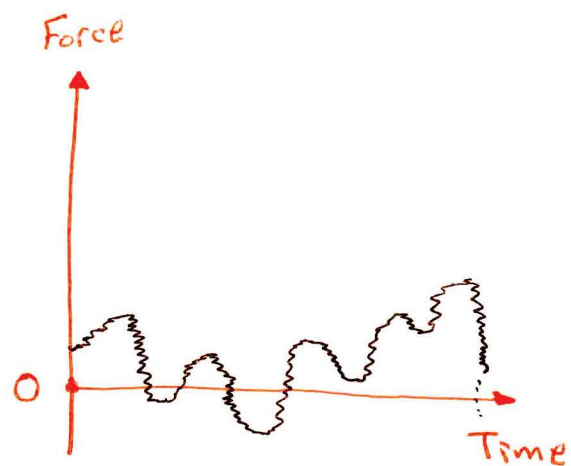
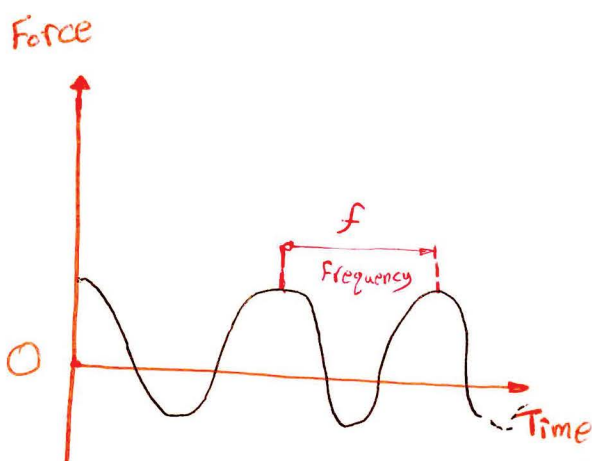
Linear Vibration if mass, damper, and spring behave Linearly

Non Linear Vibration if mass, damper, and spring behave NonLinearly

4] Deterministic and Random Vibration

Deterministic vibration if value/magnitude of excitation ^{مثير} "force or motion" acting on a vibratory system is Known at any given time

Random Vibration if value/magnitude of excitation "force or motion" acting on a vibratory system is Random



⊛ A Deterministic "periodic" excitation

⊛ A Random excitation



1.6 Vibration Analysis Procedure

* Vibratory system

↳ dynamic system with variables

↳ excitations "input"
↳ responses "output"

- Respons - - - depends on

↳ [1] The initial conditions

[2] The external conditions

Step 13- Mathematical Modeling

→ Linear
↳ or Nonlinear

انظر إلى Figure 1.16

Page 19



Step 28- Derivation of Governing Equations

- Draw Free Body Diagram, including all externally applied Forces, the reactive forces and the inertia forces
- The equations of motion of a vibrating system are usually in the form of a set of
 - ordinary differential equations for a discrete system
 - partial differential equations for a continuous system

⊛ Several approaches are commonly used to derive the governing equations.

Among them are



↳ 1. Newton's Second Law of Motion $\Sigma F = m \ddot{x}$
or $\Sigma M = J \ddot{\theta}$

≡ 2. D'Alembert's principle

≡ 3. The principle of conservation of energy

⊠ ملاحظة سيتر استخدام الطريقة الأولى "نقطة الثاني" من خلال هذا course



Step 3: Solution of the governing Equations

- we solve equation of motion
to find the response of the vibrating system
- we have the following techniques
for finding the solution
 - ⊗ Standard methods of solving differential equations
 - ⊗ Laplace transform Methods
 - ⊗ Matrix Methods
 - ⊗ Numerical Methods

Step 4: Interpretation of the Results

- The solution of the governing equations
give the displacement
velocities
and accelerations
of the various masses of the system



1.7 Spring Elements

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- The most common Type of spring is The Helical-coil spring

used in retractable pens and pencils, staplers, and suspensions of freight trucks and other vehicles

- In fact ... any elastic or deformable body or member, such as
a cable, bar
beam, shaft or plate
can be considered as a spring

- If a spring is said to be linear

Then



$$\underline{\underline{F = kx}}$$

where

k a spring constant
or spring stiffness
or spring rate

k is always positive



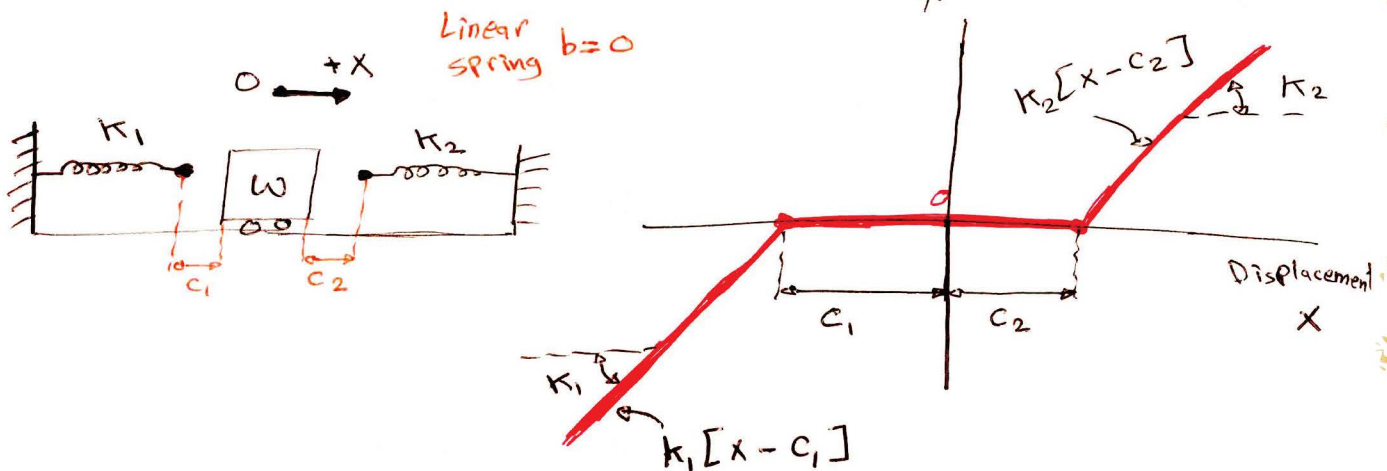
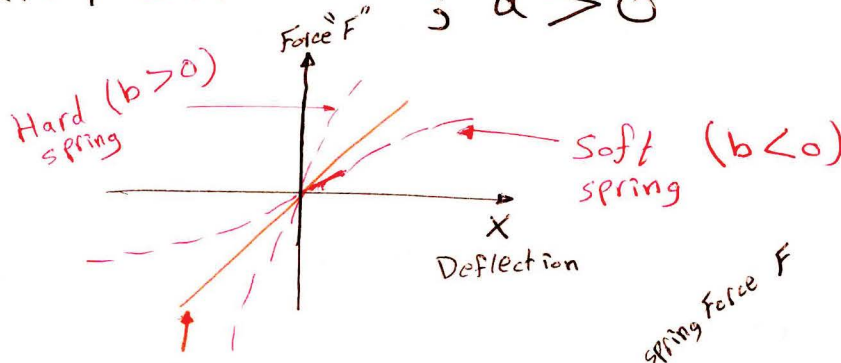
- The work done "U" in deformation a spring is stored as strain or potential Energy in the spring

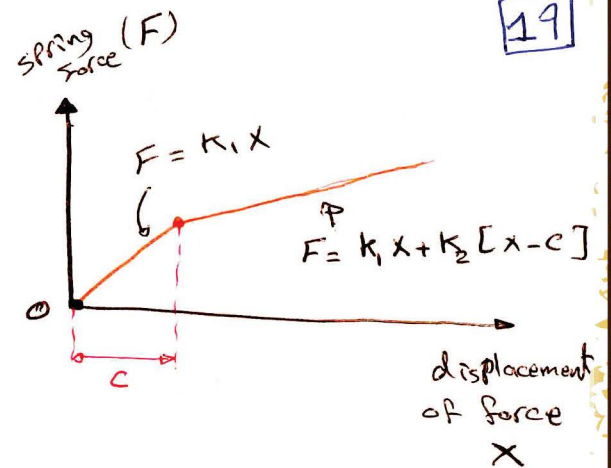
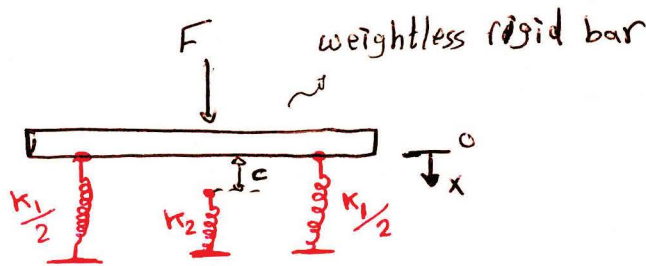
$$U = \frac{1}{2} k x^2$$

1.7.1 Nonlinear Springs

- when the deflections are large $\leadsto$ spring is nonlinear

$$F = ax + bx^3 \quad ; \quad a > 0$$



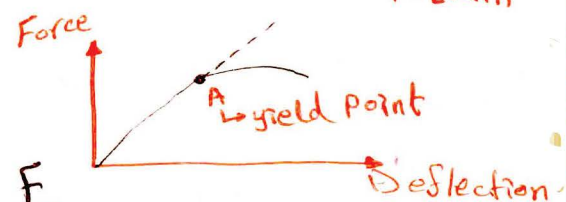
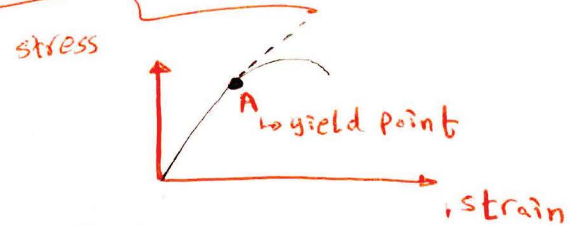


$x = 0$, no force

$\frac{k_1}{2}$, $\frac{k_1}{2}$ --- 2 Parallel spring

1.7.2 Linearization of a Nonlinear Spring

- when the stress exceeds the yield point of the material and the force-deformation relation becomes $\rightarrow$ nonlinear



⊗ Let the Static equilibrium load $\underline{F}$ acting on the spring cause a deflection of x^*

⊗ If an incremental force $\underline{\Delta F}$ is added of F ,

Then $\rightarrow$ The spring deflects by an additional quantity $\underline{\Delta x}$

(*) The new spring Force $F + \Delta F$

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can be expressed using Taylor's series expansion about the static equilibrium position x^*

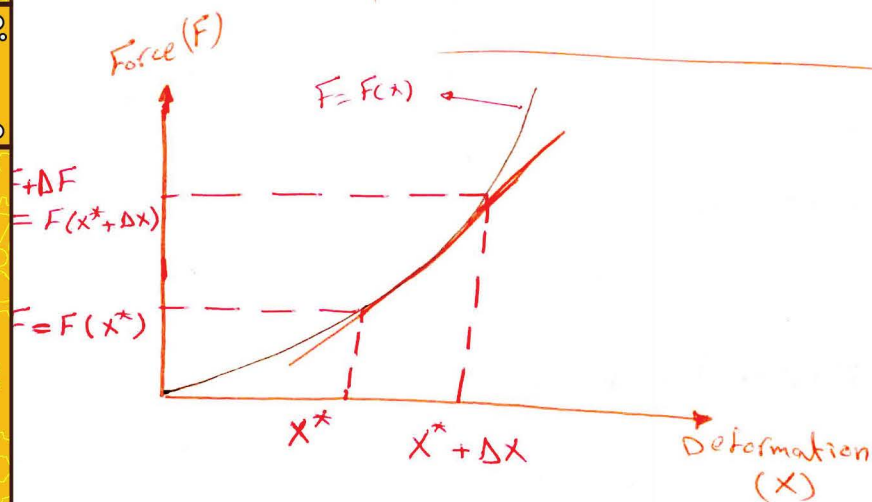
as 3-

$$F + \Delta F = F(x^* + \Delta x)$$

$$= F(x^*) + \left. \frac{dF}{dx} \right|_{x^*} \Delta x + \frac{1}{2!} \left. \frac{d^2F}{dx^2} \right|_{x^*} (\Delta x)^2 + \dots$$

For small values of Δx

$$\rightarrow F + \Delta F = F(x^*) + \left. \frac{dF}{dx} \right|_{x^*} (\Delta x)$$



(*) Since $F = F(x^*)$
 $\Delta F = k \Delta x$

k is linearized spring constant at x^*

$$\rightarrow k = \left. \frac{dF}{dx} \right|_{x^*}$$



* Ex 1.2 $w = 4,500 \text{ N}$, $F = 3,500 x + 55 x^3$ [21]

Find equivalent linearized spring constant

x in cm

F in N

Solution k^2
 $\rightarrow \boxed{x^*}$

$$F = 3,500 x + 55 x^3$$

$\downarrow$
4,500

$$\rightarrow \boxed{x_1^* = 1.2547} \quad \checkmark$$

$$x_2^* = -0.6273 + 8.0509i$$

$$x_3^* = -0.6273 - 8.0509i$$

The static equilibrium position
 given by the real root

$$\rightarrow k_{eq} = \left. \frac{dF}{dx} \right|_{x^*} = 3,500 + 165(x^*)^2$$

$$\rightarrow k_{eq} = 3,760 \text{ N/cm}$$

$$x \text{ (static deflection)} = \frac{F}{k_{eq}} = \frac{4,500}{3,760} = 1.1968 \text{ cm}$$

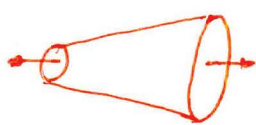
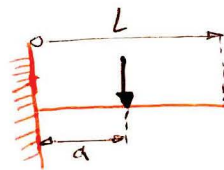
True value is $\rightarrow$ T.V $\Rightarrow F = 4,500 = 3,500 x + 55 x^3$

$$\rightarrow \boxed{x_{T.V} = 1.2547}$$



1.7.3 Spring Constant of Elastic Elements

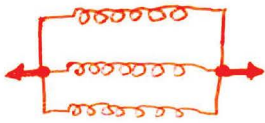
الحالة الممتدة
Keq

Shape	Description	Law
	Rod under axial load [L=Length, A= cross sectional area]	$K_{eq} = \frac{EA}{L}$
	Tapered rod under axial load [D, d :- end diameters]	$K_{eq} = \frac{\pi E D d}{4L}$
	Helical spring under axial load [d :- wire diameter D :- mean diameter n :- a number of active turns]	$K_{eq} = \frac{G d^4}{8 n D^3}$
	Fixed-Fixed beam with load at the middle	$K_{eq} = \frac{192 EI}{L^3}$
	Cantilever beam	$K_{eq} = \frac{3 EI}{L^3}$
	Simply supported beam with load at the middle	$K_{eq} = \frac{48 EI}{L^3}$

Shape

Description

Law



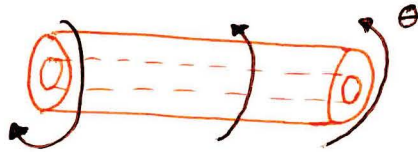
Springs in parallel

$$K_{eq} = K_1 + K_2 + \dots$$



Springs in series

$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} + \dots$$



Hollow shaft
under Torsion

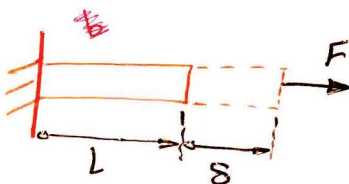
$$K_{eq} = \frac{\pi G}{32L} [D^4 - d^4]$$

[L is length
D is outer diameter
d is inner diameter]

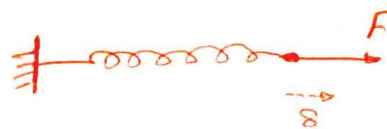
Ex 1.3 Find K_{eq}

you have
L ✓, A ✓
E ✓, F ✓

المعطيات
 $K = \frac{AE}{L}$



as spring



From strength of material

$$\sigma = \frac{F}{A} \quad (1)$$

$$\delta = \epsilon L \quad (2)$$

$$\sigma = \epsilon E \rightarrow \epsilon = \frac{\sigma}{E} \quad (3)$$

$$(3) \text{ in } (2) \rightarrow \delta = \frac{\sigma}{E} L \quad (4)$$

$$(1) \text{ in } (4) \rightarrow \delta = \frac{FL}{EA} \quad \text{or } F = k\delta \rightarrow K_{eq} = \frac{AE}{L}$$

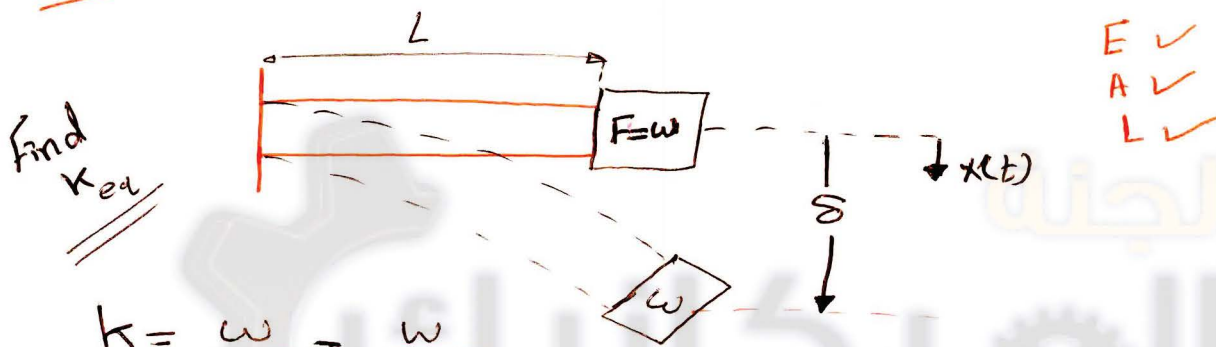


⊛ From the strength of materials

↳ The end deflection of the beam due to a concentrated load $F=w$

$$\delta = \frac{wL^3}{3EI}$$

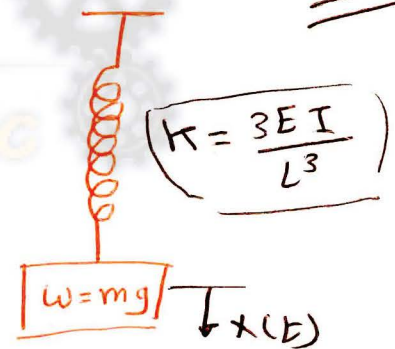
Ex 1.4 Spring constant of a Cantilever Beam



$$k = \frac{w}{\delta} = \frac{w}{\frac{wL^3}{3EI}}$$

$$\rightarrow k_{eq} = \frac{3EI}{L^3}$$

إيجاد



where

E : Young's modulus, modulus of elasticity

I : The moment of inertia of the cross section of the beam about the bending or Z-axis

[i.e., axis perpendicular to the page]

أي



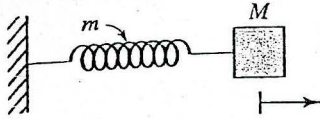
m_{eq}
 k_{eq}
 c_{eq}

جميع الحالات

25

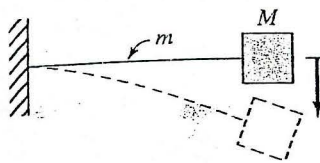
Equivalent Masses, Springs, and Dampers

Equivalent masses



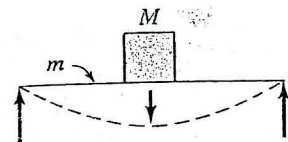
Mass (M) attached at end of spring of mass m

$$m_{eq} = M + \frac{m}{3}$$



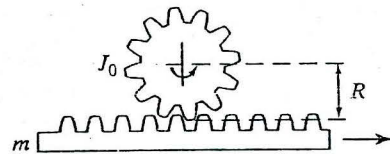
Cantilever beam of mass m carrying an end mass M

$$m_{eq} = M + 0.23 m$$



Simply supported beam of mass m carrying a mass M at the middle

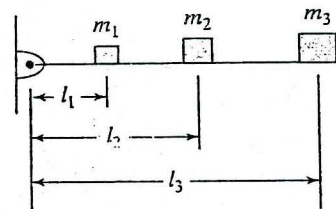
$$m_{eq} = M + 0.5 m$$



Coupled translational and rotational masses

$$m_{eq} = m + \frac{J_0}{R^2}$$

$$J_{eq} = J_0 + mR^2$$



Masses on a hinged bar

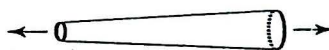
$$m_{eq1} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3$$

Equivalent springs



Rod under axial load
(l = length, A = cross-sectional area)

$$k_{eq} = \frac{EA}{l}$$



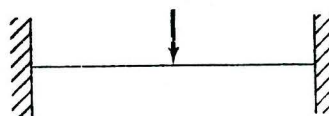
Tapered rod under axial load
(D, d = end diameters)

$$k_{eq} = \frac{\pi EDd}{4l}$$



Helical spring under axial load
(d = wire diameter,
 D = mean coil diameter,
 n = number of active turns)

$$k_{eq} = \frac{Gd^4}{8nD^3}$$



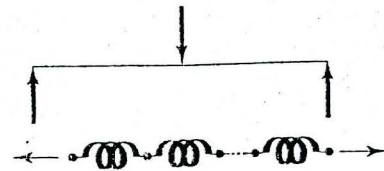
Fixed-fixed beam with load at the middle

$$k_{eq} = \frac{192EI}{l^3}$$



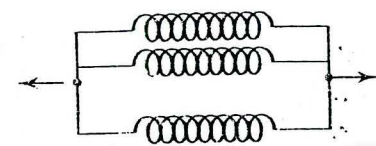
Cantilever beam with end load

$$k_{eq} = \frac{3EI}{l^3}$$



Simply supported beam with load at the middle

$$k_{eq} = \frac{48EI}{l^3}$$

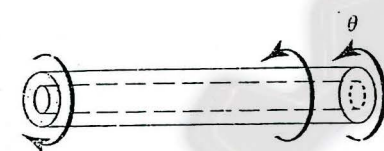


Springs in series

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \dots + \frac{1}{k_n}$$

Springs in parallel

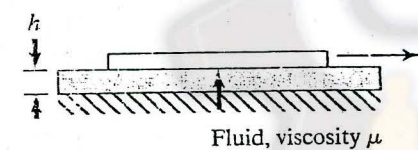
$$k_{eq} = k_1 + k_2 + \dots + k_n$$



Hollow shaft under torsion
(l = length, D = outer diameter,
 d = inner diameter)

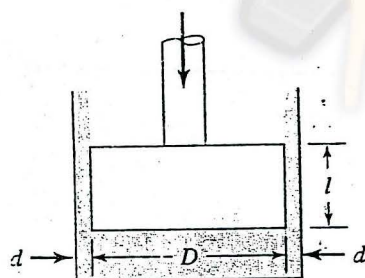
$$k_{eq} = \frac{\pi G}{32l} (D^4 - d^4)$$

Equivalent viscous dampers



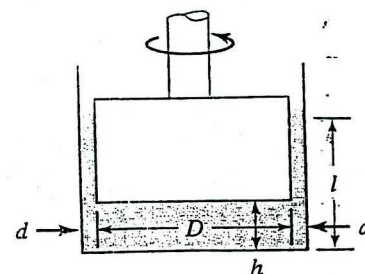
Relative motion between parallel surfaces
(A = area of smaller plate)

$$c_{eq} = \frac{\mu A}{h}$$



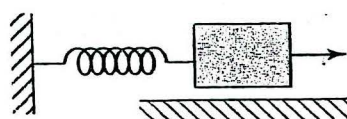
Dashpot (axial motion of a piston in a cylinder)

$$c_{eq} = \mu \frac{3\pi D^3 l}{4d^3} \left(1 + \frac{2d'}{D} \right)$$



Torsional damper

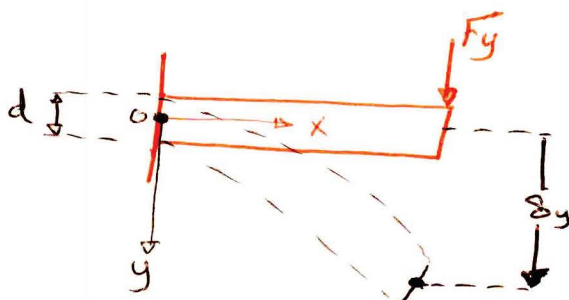
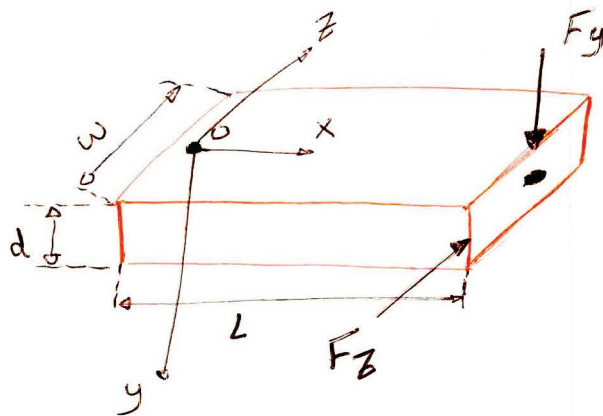
$$c_{eq} = \frac{\pi \mu D^2 (l - h)}{2d} + \frac{\pi \mu D^3}{32h}$$



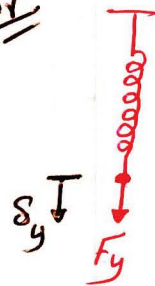
Dry friction (Coulomb damping)
(fN = friction force,
 ω = frequency,
 X = amplitude of vibration)

$$c_{eq} = \frac{4fN}{\pi \omega X}$$

Notes



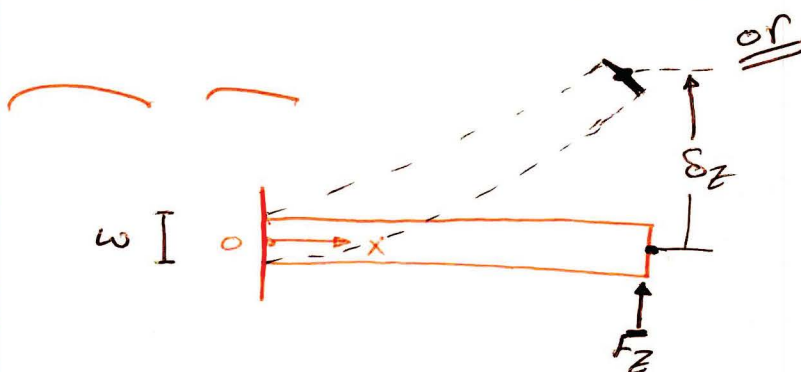
or



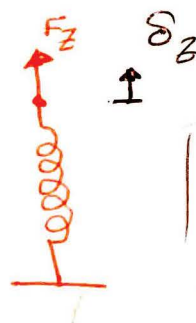
$$k_{eq} = \frac{3EI_{zz}}{L^3}$$

* When the Load is applied along the y-direction,

→ The beam bends about the Z-axis



or



$$k_{eq} = \frac{3EI_{yy}}{L^3}$$

1.8 Mass or Inertia Element

⊕ From Newton's Second Law of motion

$$F = ma$$

$$W = Fx$$

↓ The work done on mass
is stored in the Form of
the mass's kinetic energy

- For a quick and reasonably accurate
analysis o o o

The mass & damping of the beam
can be disregarded

1.8.1 Combination of Masses

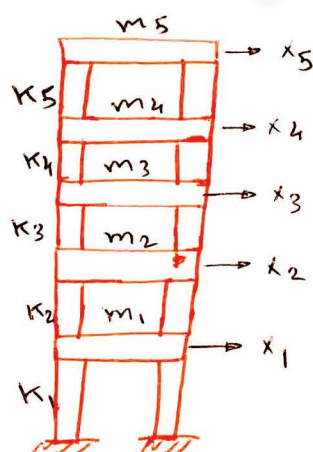


Figure a

Idealization
of a multistory
building as a
multi-degree-of-freedom
system

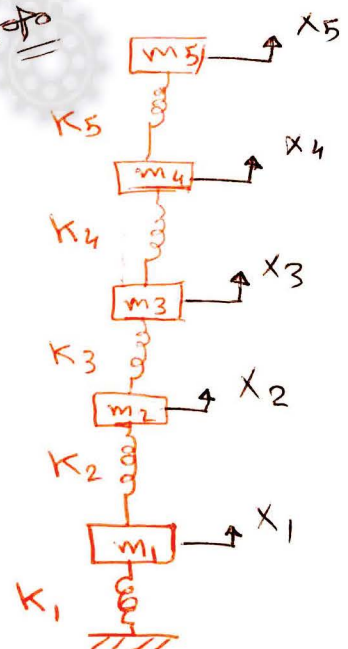


Figure b

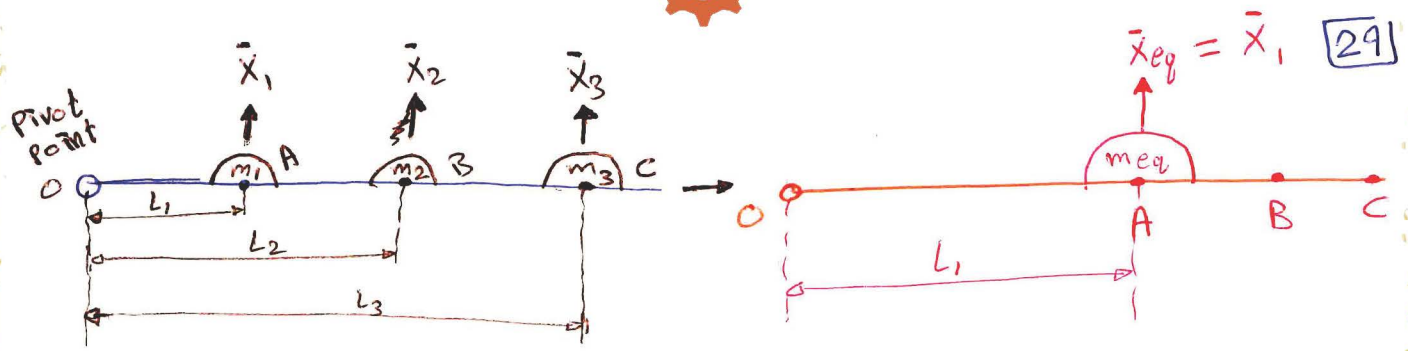


Figure 5 Translational masses connected by a rigid bar

Case1 Translation Masses connected by a rigid Bar

* The equivalent mass can be assumed to be located at any point along the bar

assume $\rightarrow \bar{x}_{eq} = \bar{x}_1$

Remember



$$x_1 = L_1 \theta$$

$$\bar{\theta} = \frac{\bar{x}_1}{L_1} = \frac{\bar{x}_2}{L_2} = \frac{\bar{x}_3}{L_3}$$

(الطاقة الحركية)

$$T_A + T_B + T_C = T_{eq}$$

الكثافة

$$\begin{cases} \bar{x}_2 = \frac{L_2}{L_1} \bar{x}_1 \\ \bar{x}_3 = \frac{L_3}{L_1} \bar{x}_1 \end{cases}$$

بدلالة $\bar{x}_1$

$$\frac{1}{2} m_1 \bar{x}_1^2 + \frac{1}{2} m_2 \bar{x}_2^2 + \frac{1}{2} m_3 \bar{x}_3^2 = \frac{1}{2} m_{eq} \bar{x}_{eq}^2$$

$$m_{eq} = m_1 + \left[\frac{L_2}{L_1} \right]^2 m_2 + \left[\frac{L_3}{L_1} \right]^2 m_3$$

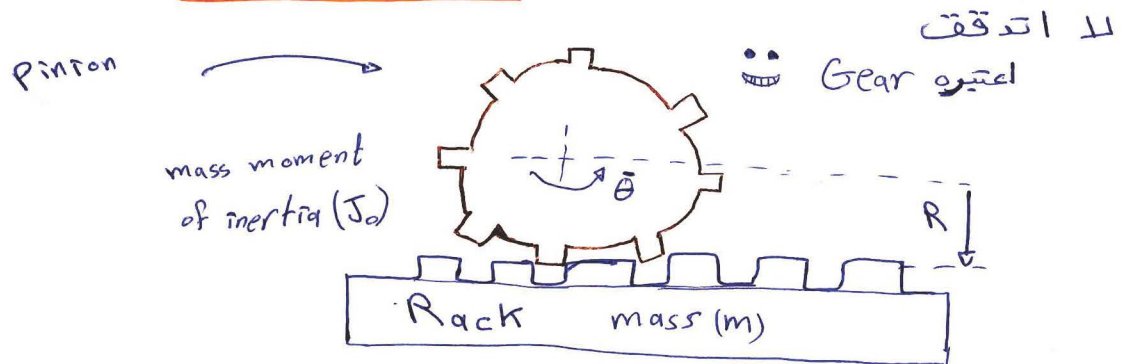
بالتعويض
والافتقار

Case 2 Translational and Rotational Masses Coupled Together

30

⊛ Let a mass "m" having a translational velocity " $\dot{x}$ " be coupled to another mass [of mass moment of inertia J_0] having a rotational velocity $\dot{\theta}$

as in the rack-and-pinion arrangement shown in Figure



⊛ These Two masses can be Combined to obtain either **و!**

- (1) m_{eq} a single equivalent translational mass
- (2) J_{eq} a single equivalent Rotational mass

(1) m_{eq} $\dot{x}_{eq} = \dot{x}$

$$\frac{1}{2} m_{eq} \dot{x}_{eq}^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$x = \theta R \rightarrow \left(\dot{\theta} = \frac{\dot{x}}{R} \right)$$

تعويض واختصار

$$\rightarrow m_{eq} = m + \frac{J_0}{R^2}$$

(2) J_{eq} $\dot{\theta}_{eq} = \dot{\theta}$

$$\frac{1}{2} J_{eq} \dot{\theta}_{eq}^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$x = \theta R \rightarrow \left(\dot{x} = \dot{\theta} R \right)$$

تعويض واختصار

$$J_{eq} = J_0 + m R^2$$

(OR)



بالتالي

Ex 1.12 &
Page 44

Ex 1.11
Page 43

المعلم على (*)
(مف)

1.10 Harmonic Motion

⊠ Oscillatory motion

may → repeat itself regularly
such as Simple Pendulum
(OR) → display considerable irregularity
such as ground motion during an earthquake

⊠ Periodic Motion

If the motion is repeated after
equal intervals of time

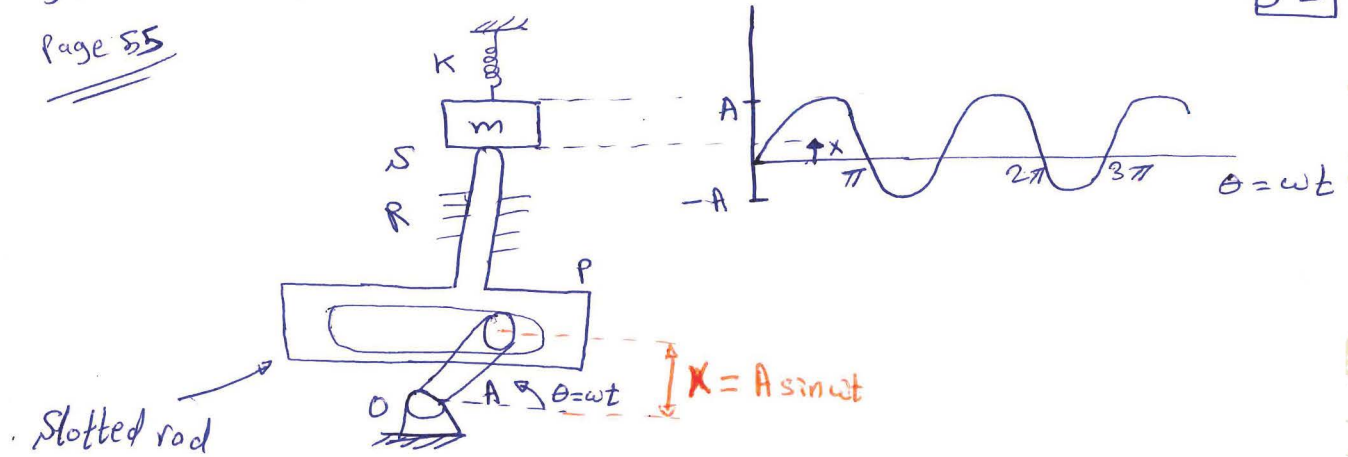
- The simplest type of periodic motion
is → Harmonic motion



Figure 8-Scotch yoke mechanism

Page 55

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reciprocates in the vertical guide R

✳ The motion given by $x = A \cos \omega t$ is another example of simple Harmonic motion

✳ For vectorial representation of Harmonic motion

Figure 1-41, Page 56

$$y = A \sin \omega t$$

$$x = A \cos(\omega t)$$

$$v = \dot{x} = \omega A \sin[\omega t + 90^\circ]$$

$$a = \ddot{x} = \omega^2 A \cos[\omega t + 180^\circ]$$

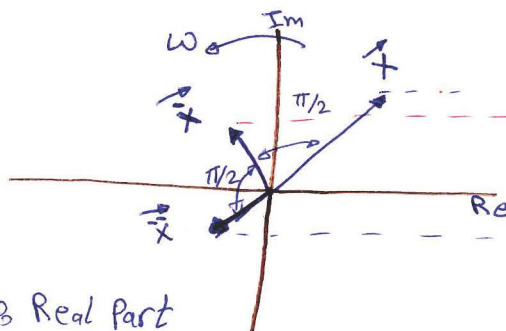
$$x = A \sin[\omega t]$$

$$\dot{x} = \omega A \cos[\omega t + 90^\circ]$$

$$\ddot{x} = \omega^2 A \sin[\omega t + 180^\circ]$$

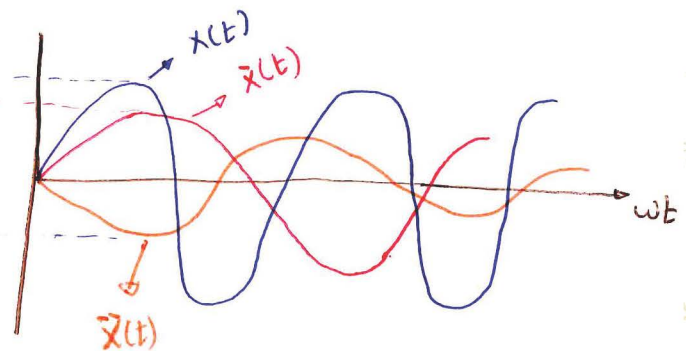
✳ $\vec{a}$ [vector] Leads the $\vec{v}$ by 90°

✳ $\vec{v}$ [vector] Leads the $\vec{x}$ by 90°



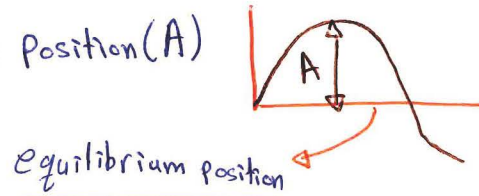
Re: Real Part

Im: Imaginary part



1.10.5 Definitions and Terminology

✱ **Amplitude** :- The max displacement of a vibrating from its equilibrium position (A)



✱ **Period of oscillation** :- The Time taken to complete one cycle of motion
[Time period]

$$T = \frac{2\pi}{\omega} \quad \text{where } \omega \text{ is circular frequency or angular velocity rad/s}$$

✱ **frequency of oscillation** :- The number of cycles per unit time
[Linear frequency / frequency]

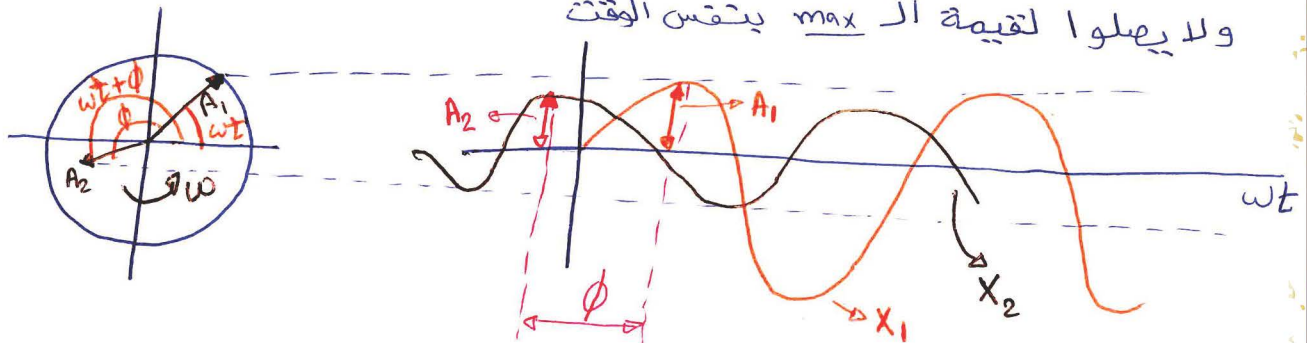
$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{cycle second} \quad [Hz] \quad (\text{الهيرتز})$$

✱ **phase angle "φ"**

$$x_1 = A_1 \sin[\omega t] \quad , \quad x_2 = A_2 \sin[\omega t + \phi]$$

* The two harmonic motion $[x_1 \& x_2]$ are called Synchronous because $\rightarrow$ They have the same frequency " ω "

* Two synchronous oscillations $\rightarrow$ لا يلزم أن يكونوا بنفس الـ A ولا يهلوا لقيمة الـ $\max$ بنفس الوقت



* Natural Frequency

If a system, after an initial disturbance, is left to vibrate on its own, The frequency with which it oscillates with out external Forces is known as its natural Frequency

- A vibratory system having n degrees of freedom will have, in general, n distinct natural frequencies of vibration ~~as~~ As will be seen in CH 5

* Beats When two harmonic Motions, with frequencies close to one another, are added, the resulting motion exhibits a phenomenon known as beats

- For example if $\rightarrow x_1(t) = X \cos[\omega t]$
 $x_2(t) = X \cos[\omega + \delta] t$

where δ is a small quantity

$$x(t) = x_1 + x_2 \leadsto$$

$$x(t) = \underbrace{2X \cos\left[\frac{\delta t}{2}\right]}_{\text{Varying amplitude}} \underbrace{\cos\left[\omega + \frac{\delta}{2}\right] t}_{\text{Frequency}}$$

وبتطبيق متطابقات
تنتج العلاقة



(*) Whenever --- the amplitude reaches a maximum it is called a beat

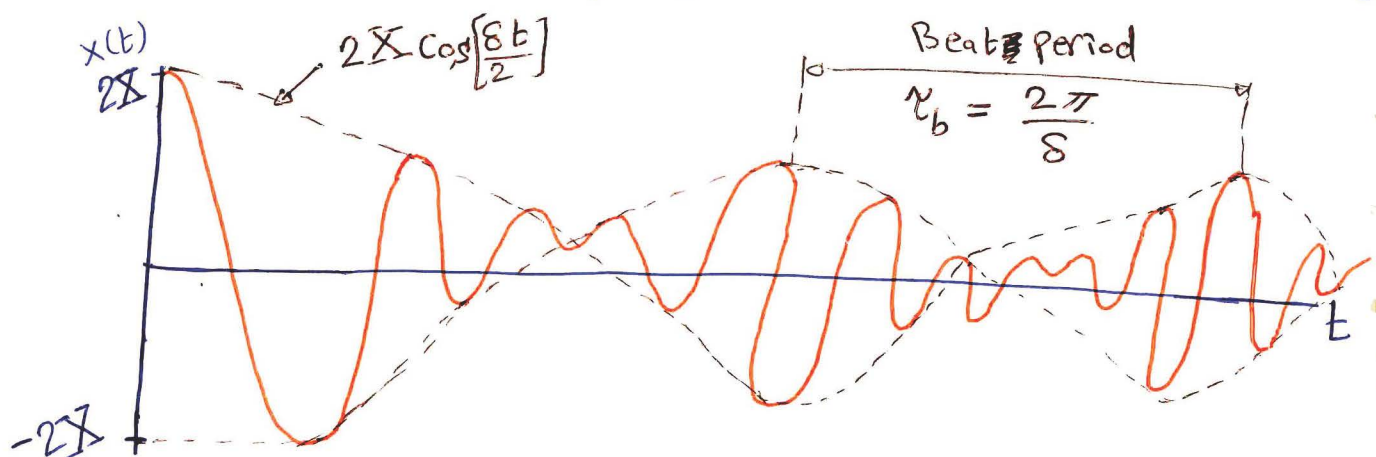
35

(*) The frequency $[\delta]$ at which the amplitude builds up and dies down between 0 and $2X$ is known as beat frequency

(*) The Beating phenomenon occurs

when $\rightarrow$ The forcing frequency " ω " is close to

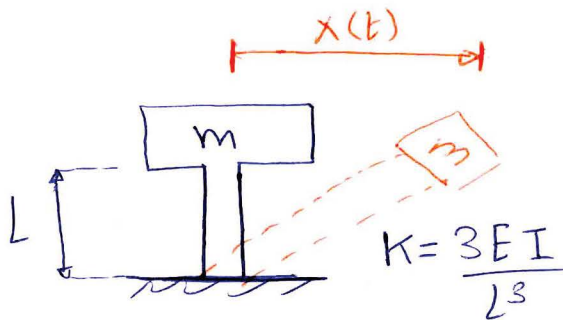
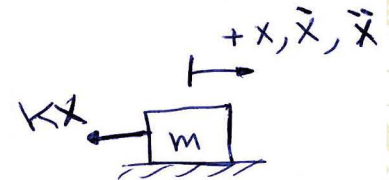
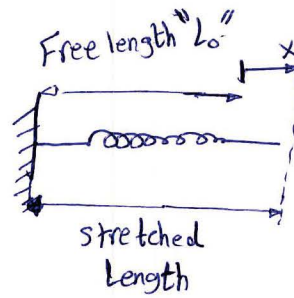
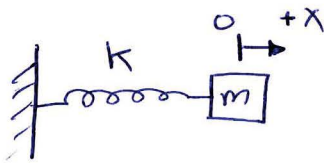
The natural frequency " ω_n " [see section 3.3.2]



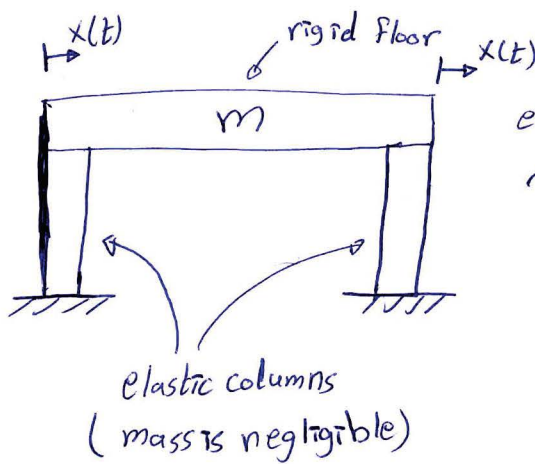
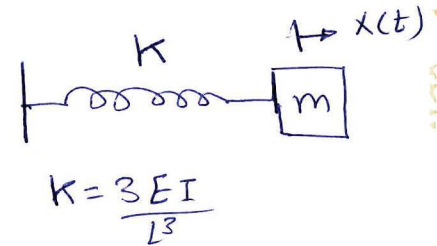
Figures: phenomenon of beats

CH2 Free Vibration of Single-Degree-of-Freedom Systems

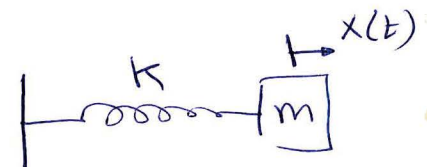
2.1 Introduction



equivalent spring-mass



equivalent spring-mass



2.2 Free Vibration of an Undamped Translational System

2.2.1 Equation of motion using Newton's second law of motion

⊠ The rate of change of momentum of mass is equal to the force acting on it

$$F = ma = m \frac{dv}{dt}$$

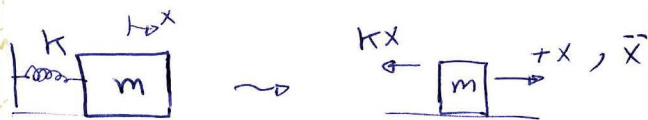
$$\rightarrow \vec{F}(t) = \frac{d}{dt} \left[m \frac{d\vec{x}(t)}{dt} \right] = m \frac{d^2\vec{x}(t)}{dt^2} = m \ddot{\vec{x}}$$

بالتفصيل

Translatory motion $\rightarrow \vec{F}(t) = m \ddot{\vec{x}}$

Rotational motion $\rightarrow \vec{M}(t) = J \ddot{\vec{\theta}}$





$$m\ddot{x} + Kx = 0 \quad \text{eqn. 2.3}$$

Laplace $\rightarrow$

$$mS^2 + K = 0 \rightarrow \text{Auxiliary or characteristic equation corresponding to eqn. 2.3}$$

$$S = \pm i\omega_n \rightarrow \text{two values of } S$$

Known 1 roots of characteristic eqn.

2 the eigenvalues

3 The characteristic values of the problem

without Force
Free Vibration

without ζ
Undamped

x not θ
Linear

* The equation $\rightarrow m\ddot{x} + Kx = 0$

* The general solution $\rightarrow X(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$

where $A_1 = x_0$, $A_2 = \frac{\dot{x}_0}{\omega_n}$

or $\rightarrow X(t) = A \cos[\omega_n t - \phi]$

where $A = \sqrt{A_1^2 + A_2^2}$, $\phi = \tan^{-1} \frac{A_2}{A_1}$

or $\rightarrow X(t) = A_0 \sin[\omega_n t + \phi_0]$

where $A_0 = \sqrt{A_1^2 + A_2^2}$, $\phi_0 = \tan^{-1} \frac{A_1}{A_2}$



* Some laws

$$\omega_n = 2\pi f_n$$

circular
natural
frequency

natural
frequency

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ملاحظة

لذلك يطلب
natural freq.

لـ ω_n أيها ٥٥٥ لأنه قد يقسمها ونسبها التعبير

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$$

$$\tau_n = \frac{1}{f_n}$$

natural
period

2.3 Free Vibration of an Undamped Torsional System

* Undamped Torsional [Free]

- Some laws

$$m\ddot{x} + kx = 0$$

نكافئ $\rightarrow J_0 \ddot{\theta} + k_t \theta = 0$

J_0 is polar mass moment of inertia

$$m \rightsquigarrow J_0$$

$$x \rightsquigarrow \theta$$

$$k \rightsquigarrow k_t$$

$$M_t = \frac{GJ}{L} \theta$$

where M_t is torque

θ is angular rotation

G is shear modulus, L is length of shaft

$$J_0 = \frac{\pi(d_{out}^4 - d_{in}^4)}{32}$$

where

J_0 is polar moment of inertia

Hollow shaft



if solid shaft $d_{in} = 0$

$$\omega_n = \sqrt{\frac{k_t}{J_0}}$$

$$k_t = \frac{M_t}{\theta} = \frac{GJ}{L}$$

where

k_t is torsional spring constant



* Polar mass moment of inertia " J_o "
of a disk

$$J_o = \frac{\rho h \pi D^4}{32} = \frac{W D^2}{8g}$$

where ρ is mass density, W is weight of disk
 h is thickness, D is diameter of disk

Free Vibration Undamped $\rightarrow \theta$ not x
Torsion

* The general ~~solution~~ ^{equation} $\rightarrow J_o \ddot{\theta} + K_t \theta = 0$

* The general solution $\rightarrow \theta[t] = A_1 \cos \omega_n t + A_2 \sin \omega_n t$

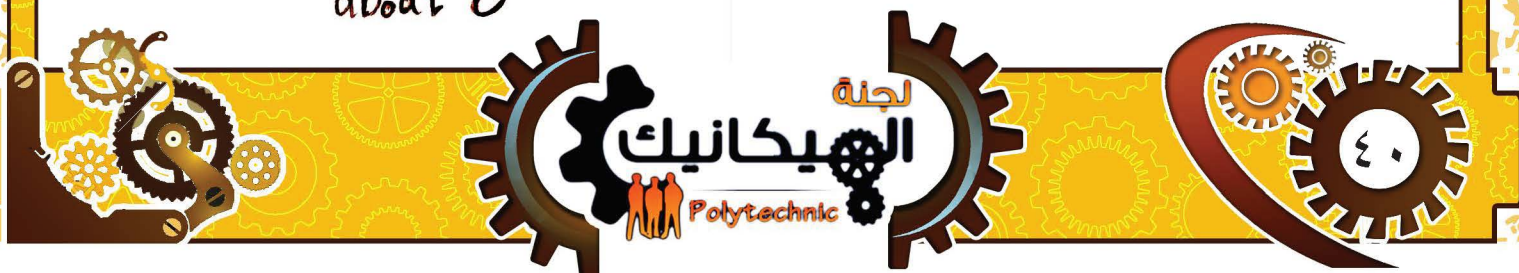
where $A_1 = \theta_o$, $A_2 = \frac{\dot{\theta}_o}{\omega_n}$

* Some Laws

$$\omega_n = \sqrt{\frac{K_t}{J_o}} = \sqrt{\frac{g}{\delta_{st}}}$$

$$J_o = m k_o^2$$

where k_o is radius of gyration about O



2.5 Rayleigh's Energy Method

$$T_1 + U_1 = T_2 + U_2$$

where T is Kinetic Energy
 U is Potential Energy

1 $\rightarrow$ Time when mass is passing through its static equilibrium position

$$\therefore U_1 = 0$$

2 $\rightarrow$ Time corresponding to max displacement of mass

$$\therefore T_2 = 0$$

$$T_1 + 0 = 0 + U_2 \rightarrow T_{\max} = U_{\max} \quad \text{rayleigh}$$

Ex 2.8 -- Page 155

⊗ يوجد مثال على طريقة Rayleigh

لكن غالبا لا نطلب للحل لأننا نعلم في هذا course طريقة قانون نيوتن الثاني

$$F = m\ddot{x}$$

Polytechnic



2-6 Free vibration with viscous Damping

Free Damped Linear

equation $\rightarrow m\ddot{x} + c\dot{x} + kx = 0$

General Solution

Underdamped

$$x[t] = e^{-\zeta\omega_n t} [C_1 \cos\omega_d t + C_2 \sin\omega_d t]$$

$$C_1 = X_0, \quad C_2 = \frac{\dot{X}_0 + \zeta\omega_n X_0}{\omega_d}$$

(OR) $x[t] = X_0 e^{-\zeta\omega_n t} \sin[\omega_d t + \phi_0]$

$$X_0 = \sqrt{C_1^2 + C_2^2}, \quad \phi_0 = \tan^{-1} \frac{C_1}{C_2}$$

(OR) $x[t] = \bar{X} e^{-\zeta\omega_n t} \cos[\omega_d t - \phi]$

$$\bar{X} = \sqrt{C_1^2 + C_2^2}, \quad \phi = \tan^{-1} \frac{C_2}{C_1}$$

Overdamped

$$x[t] = C_1 e^{[-\zeta + \sqrt{\zeta^2 - 1}]\omega_n t} + C_2 e^{[-\zeta - \sqrt{\zeta^2 - 1}]\omega_n t}$$

$$C_1 = \frac{X_0 \omega_n [\zeta + \sqrt{\zeta^2 - 1}] + \dot{X}_0}{2 \omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-X_0 \omega_n [\zeta - \sqrt{\zeta^2 - 1}] - \dot{X}_0}{2 \omega_n \sqrt{\zeta^2 - 1}}$$

Critically damped

$$x[t] = e^{-\zeta\omega_n t} [C_1 + C_2 t]$$

$$C_1 = X_0, \quad C_2 = \dot{X}_0 + \omega_n X_0$$

For Torsion

$$J_0 \ddot{\theta} + C_t \dot{\theta} + K_t \theta = 0$$

و المعادلات نفسها مع مراعاة تبديل
 $m, x, c, k \rightarrow J_0, \theta, C_t, K_t$ على التوالي

⊠ Important Law $\rightarrow \zeta = \frac{C_{eq}}{C_{cr}} = \frac{C_{eq}}{2\sqrt{K_{eq} m_{eq}}}$

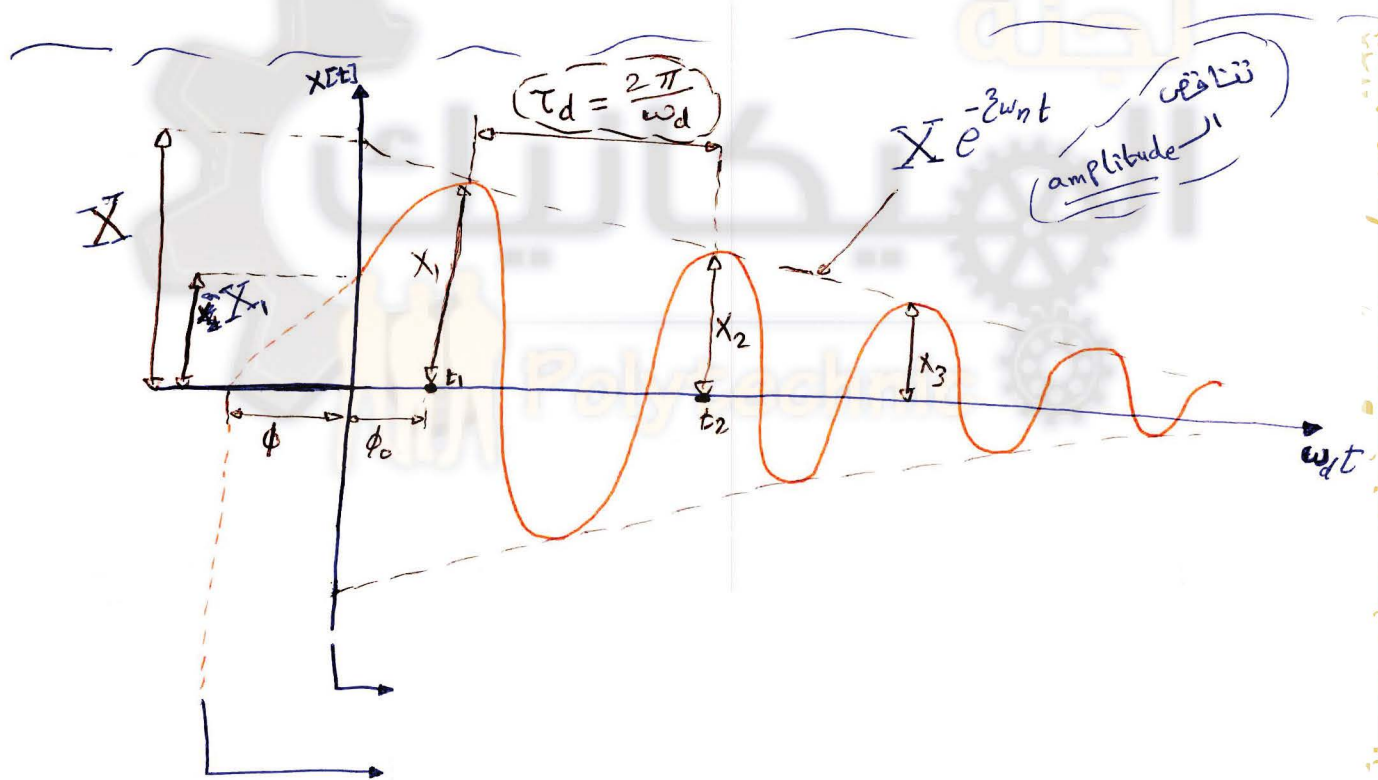
الكافئة equivalent

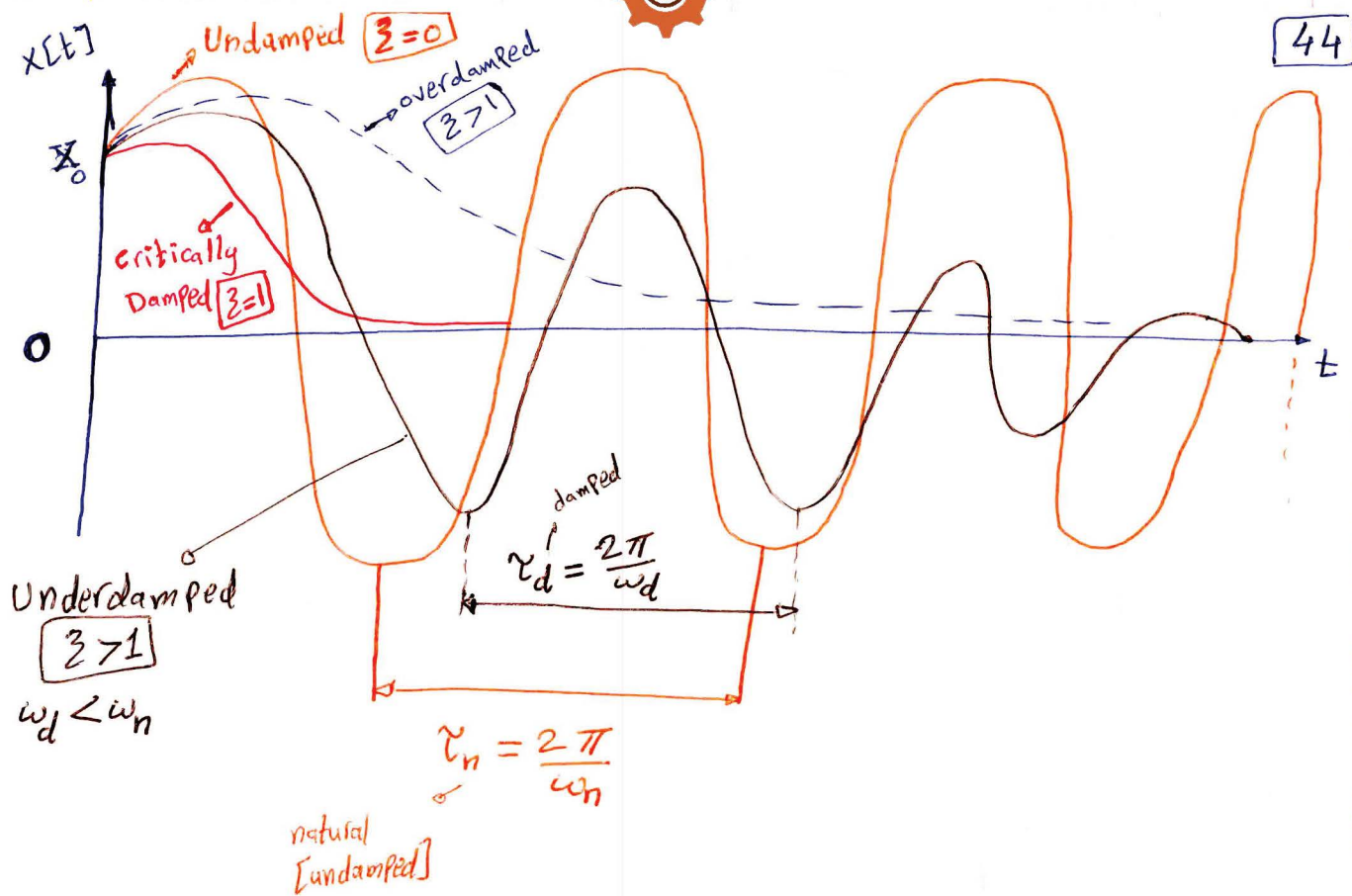
When $\zeta = 0 \rightarrow$ Undamped, $\zeta > 1 \rightarrow$ overdamped

$\zeta = 1 \rightarrow$ critical Damped, $\zeta < 1 \rightarrow$ underdamped

$\omega_n = \sqrt{\frac{K_{eq}}{m_{eq}}}$ or Go to Page 25, 26 $\leftarrow \begin{cases} K_{eq} \\ m_{eq} \end{cases}$

$C_{eq} = 2\zeta\omega_n m_{eq}$ or Go to Page 26 $\leftarrow C_{eq}$





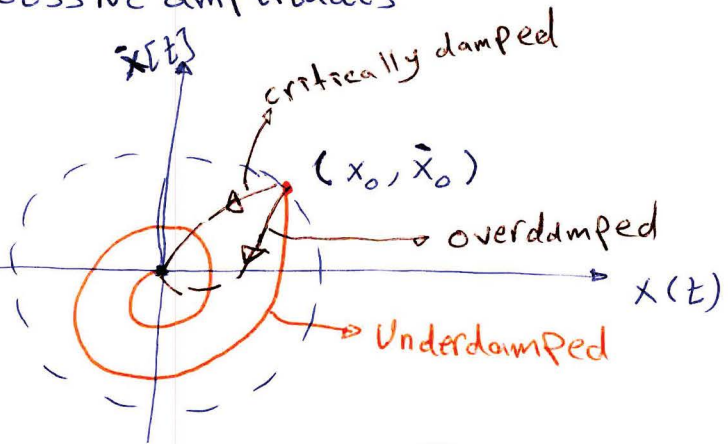
(*) Comparison of motions with different types of damping

- Underdamped $\rightarrow z < 1$, $C_{eq} < C_{cr}$
 - Critically damped $\rightarrow z = 1$, $C_{eq} = C_{cr}$
 - Overdamped $\rightarrow z > 1$, $C_{eq} > C_{cr}$
- $$z = \frac{C_{eq}}{C_{cr}}$$

Logarithmic Decrement

Defines: The natural Logarithm of the ratio of any two successive amplitudes

Phase plane



(*) معلومة التناقص في ال Amplitude يكون فقط في نظام ال Underdamped

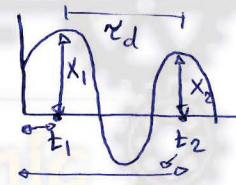
نظيقت على

Logarithmic Decrement

لذلك لا يجاد ال

$$x(t) = x_0 e^{-\zeta \omega_n t} \cos[\omega_d t - \phi_0]$$

$$\frac{x_1}{x_2} = \frac{x_0 e^{-\zeta \omega_n t_1} \cos[\omega_d t_1 - \phi_0]}{x_0 e^{-\zeta \omega_n t_2} \cos[\omega_d t_2 - \phi_0]}$$



$$\tau_d = t_2 - t_1$$

من الشكل

$$t_2 = t_1 + \tau_d \quad \text{--- (1)}$$

$$\tau_d = \frac{2\pi}{\omega_d} \quad \text{--- (2)}$$

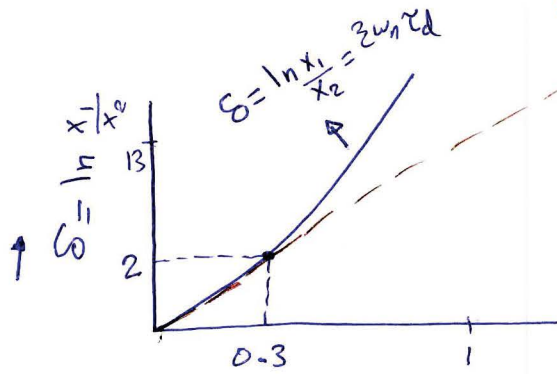
بعد تعويض (2) في (1) وتوحيد المقامات

$$\cos[\omega_d t_2 - \phi_0] = \cos[2\pi + \omega_d t_1 - \phi_0] = \cos[\omega_d t_1 - \phi_0]$$

$$\frac{x_1}{x_2} = \frac{e^{-\zeta \omega_n t_1}}{e^{-\zeta \omega_n [t_1 + \tau_d]}} = e^{\zeta \omega_n \tau_d}$$

$$\delta = \ln \frac{x_1}{x_2} = \zeta \omega_n \tau_d$$

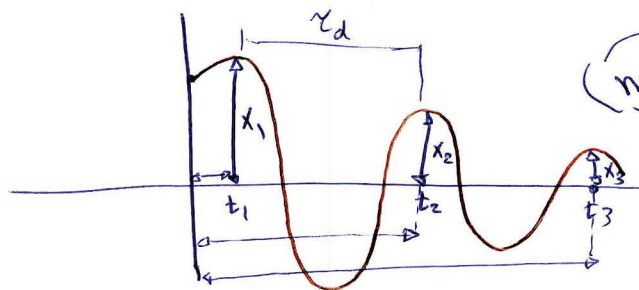




$$\delta \approx 2\pi z \quad \text{if } z \ll 1$$

$$z = \frac{C_{eq}}{C_{cr}}$$

(*) يرجى التمييز بين الـ 2 curves عند $z = 0.3$



تلقين

$$\left(\tau_d = \frac{2\pi}{\omega_d} \right) \quad \text{also} \quad \left(\tau_d = t_2 - t_1 \right)$$

☐ if z is known $\rightarrow$ $\delta = \frac{2\pi z}{\sqrt{1-z^2}}$

☐ if δ is known $\rightarrow$ $z = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$

☐ if z is not known $\rightarrow$

$$\delta = \ln \frac{x_{n-1}}{x_n}$$

$$\delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}$$

amplitude متتالية والتي بعدها

بين الـ amplitude الأولى والأخيرة

* Torsional system with viscous damping

$$T = -C_t \dot{\theta}$$

where

T : viscous damping Torque

C_t : torsional viscous damping

$\dot{\theta}$: angular velocity

* negative sign denotes that the damping torque is opposite to the direction of angular velocity

$$J_o \ddot{\theta} + C_t \dot{\theta} + K_t \theta = 0$$

where K_t : Spring constant

[restoring torque per unit angular displacement]

$$\omega_n = \sqrt{\frac{K_t}{J_o}}$$

$$\zeta = \frac{C_{teq}}{C_{cr}} = \frac{C_{teq}}{2\sqrt{K_{teq} J_{oeq}}}$$

linear

* لكل فيها كحل 1

$m \sim J_o$

$x \sim \theta$

$c \sim C_t$

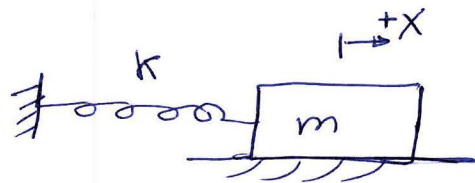
$k \sim K_t$

مع مراعاة تحويل

2.4 Free Vibration with Coulomb Damping

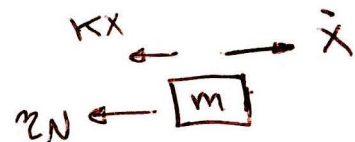
Coulomb or Dry Friction dampers

قوة الاحتكاك
 $F = \mu N$
 where N is normal force



Case 1 ^{شوط الازطاب} From [left → right]

(eqn.) $\rightarrow m\ddot{x} + Kx + \mu N = 0$

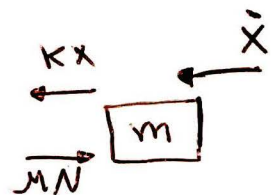


(General solution) $\rightarrow x[t] = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu N}{K}$

where $-A_1 = -x_0 + \frac{3\mu N}{K}$, $A_2 = 0$

Case 2 ^{شوط الازطاب} From [left ← right]

(eqn.) $\rightarrow m\ddot{x} + Kx - \mu N = 0$



(General solution) $\rightarrow x[t] = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu N}{K}$

where $A_3 = x_0 - \frac{\mu N}{K}$, $A_4 = 0$



Linear

r is number of half cycles to stop { For Torsional System

$$r \geq \left\lceil \frac{x_0 - \frac{MN}{K}}{\frac{2MN}{K}} \right\rceil$$

$$r \geq \left\lceil \frac{\theta_0 - \frac{T}{K_t}}{\frac{2T}{K_t}} \right\rceil$$

$$r = \frac{n}{2}, \quad t = n\tau$$

$$X = x_0 - n * \frac{4MN}{K}$$

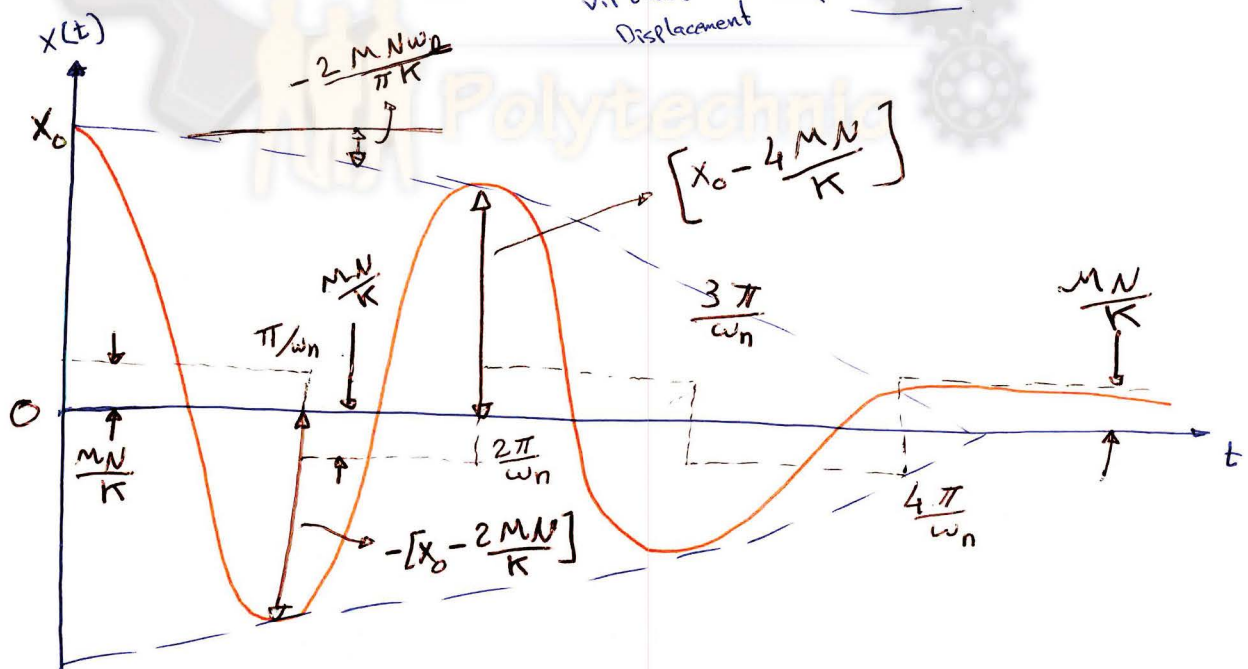
$$\theta_r = \theta_0 - r * \frac{2T}{K_t}$$

[*] The Term $\frac{MN}{K}$ is a constant representing the virtual Displacement of the spring under the force MN if it were applied as a static force

spring force $F_s = KX \Rightarrow MN = KX \leadsto X = \frac{MN}{K}$

static Force

virtual Displacement



[*] Motion of the mass with coulomb damping

☒ تلخيص قوانين CH1 & CH2 المهمة للحد 000

Free Undamped Linear

☒ $m\ddot{x} + kx = 0$

→ $x[t] = A_1 \cos \omega_n t + A_2 \sin \omega_n t$

where $A_1 = x_0$, $A_2 = \frac{\dot{x}_0}{\omega_n}$

OR → $x[t] = A \cos[\omega_n t - \phi]$

where $A = \sqrt{A_1^2 + A_2^2}$, $\phi = \tan^{-1} \frac{A_2}{A_1}$
↳ amplitude

OR → $x[t] = A_0 \sin[\omega_n t + \phi_0]$

where $A_0 = \sqrt{A_1^2 + A_2^2}$, $\phi_0 = \tan^{-1} \frac{A_1}{A_2}$
↳ amplitude

Free Undamped Torsion

☒ $J_0 \ddot{\theta} + k_t \theta = 0$ 000 Same Solution

but $m \rightsquigarrow J_0$,

$x \rightsquigarrow \theta$, $k \rightsquigarrow k_t$



Free Damped linear

★ $m\ddot{x} + C\dot{x} + Kx = 0$

Underdamped $\zeta < 1$ $\rightarrow x[t] = e^{-\zeta\omega_n t} [c_1 \cos \omega_d t + c_2 \sin \omega_d t]$

where $c_1 = x_0$
 $c_2 = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}$

OR $\rightarrow x[t] = \bar{X}_0 e^{-\zeta\omega_n t} \sin[\omega_d t + \phi_0]$

where $\bar{X}_0 = \sqrt{c_1^2 + c_2^2}$, $\phi_0 = \tan^{-1} \frac{c_1}{c_2}$

OR $\rightarrow x[t] = \bar{X} e^{-\zeta\omega_n t} \cos[\omega_d t - \phi]$

where $\bar{X} = \sqrt{c_1^2 + c_2^2}$, $\phi = \tan^{-1} \frac{c_2}{c_1}$

Overdamped $\zeta > 1$ $\rightarrow x[t] = c_1 e^{[-\zeta + \sqrt{\zeta^2 - 1}]\omega_n t} + c_2 e^{[-\zeta - \sqrt{\zeta^2 - 1}]\omega_n t}$

where $c_1 = \frac{x_0 \omega_n [\zeta + \sqrt{\zeta^2 - 1}] + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$

$c_2 = \frac{-x_0 \omega_n [\zeta - \sqrt{\zeta^2 - 1}] - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$

Critically damped $\zeta = 1$ $\rightarrow x[t] = e^{-\zeta\omega_n t} [c_1 + c_2 t]$

where $c_1 = x_0$, $c_2 = \dot{x}_0 + \omega_n x_0$



Free Damped Torsion

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$$[*] J_0 \ddot{\theta} + C_t \dot{\theta} + K_t \theta = 0$$

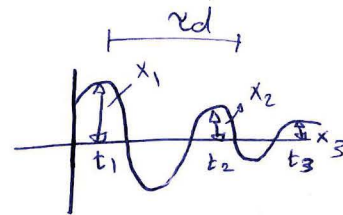
Same solution as linear

but $m \rightarrow J_0$, $C \rightarrow C_t$
 $x \rightarrow \theta$, $K \rightarrow K_t$

[*] Logarithmic Decrement

$$[*] \delta = \frac{2\pi \zeta}{\sqrt{1-\zeta^2}}$$

$$[*] \zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$



$$\begin{cases} x_d = t_2 - t_1 = t_3 - t_2 \\ x_d = \frac{2\pi}{\omega_d} \end{cases}$$

$$[*] \delta = \ln \frac{x_{n-1}}{x_n} , \quad [*] \delta = \frac{1}{n} \ln \frac{x_1}{x_{n+1}}$$

For Torsional Motion

$$[*] M_t = \frac{GJ}{L} \theta$$

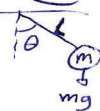
(for Disk)

$$[*] J_0 = \frac{\rho h \pi D^4}{32} = \frac{W D^2}{8g}$$

[kg.m²]

$$[*] K_t = \frac{M_t}{\theta} = \frac{GJ}{L}$$

$$[*] K_t = mgL \quad \text{for Pendulum}$$



$$[*] J = \frac{\pi (d_o^4 - d_i^4)}{32}$$

[m⁴]

For Hollow shaft
 if we have solid shaft
 $d_i = 0$

$$[*] \sum M[t] = J_0 \ddot{\theta}$$

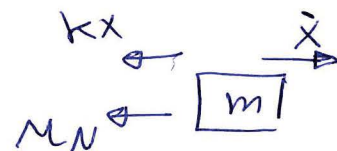
$$[*] T = -C_t \dot{\theta}$$



Free Coulomb damping "dry friction"

Case 1 $\boxed{*} m\ddot{x} + kx + MN = 0$

شوط الزفان

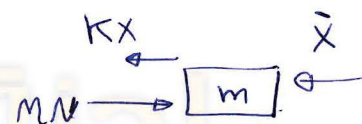


Solution $\rightarrow x[t] = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{MN}{k}$

where $-A_1 = -x_0 + \frac{3MN}{k}, A_2 = 0$

Case 2 $\boxed{*} m\ddot{x} + kx - MN = 0$

شوط اليا



Solution $\rightarrow x[t] = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{MN}{k}$

where $A_3 = x_0 - \frac{MN}{k}, A_4 = 0$

Linear

r number of half cycles to stop

$$r \geq \left\lceil \frac{x_0 - \frac{MN}{k}}{\frac{2MN}{k}} \right\rceil$$

$$r = \frac{n}{2}, t = n\tau$$

$$x = x_0 - n * \frac{4MN}{k}$$

For Torsional System

$$r \geq \left\lceil \frac{\theta_0 - \frac{T}{k_t}}{\frac{2T}{k_t}} \right\rceil$$

$$\theta_r = \theta_0 - r * \frac{2T}{k_t}$$



✱ هذه الصفحة تركتها لك

لإضافة ما تشاء من ملاحظات وقوانين



CH3 Harmonically Excited Vibration

3.1 Introduction

☒ The response of a system to a harmonic excitation is called **Harmonic Response**

☒ The response of a dynamic system to suddenly applied nonperiodic excitations is called **Transient Response**

☒ Dynamic response of a single-degree-of-freedom system under harmonic excitations of the form

$$F[t] = F_0 \cos[\omega t + \phi]$$

or $F[t] = F_0 \sin[\omega t + \phi]$

where

F_0 : The amplitude

ω : frequency

ϕ : phase angle $\circ \circ \circ$ depends on value of $F(t)$
at $t=0$, usually taken to be zero

⊛ Resonance $\rightarrow \omega = \omega_n \rightarrow r = \frac{\omega}{\omega_n} = 1$

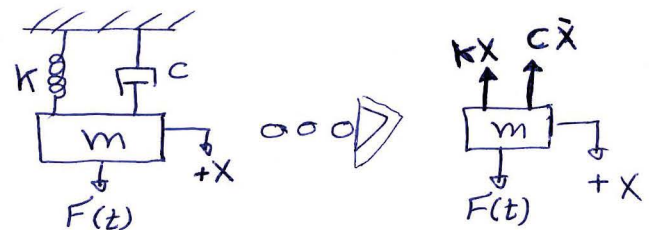
$\rightarrow$ makes failure

Page 261

harmonic excited vibration

* أمثلة على

3.2 Equation of Motion



$$[*] \quad m \ddot{x} + c \dot{x} + kx = F(t)$$

general solution $\leadsto x[t] = x_h + x_p$

where

x_h is Homogeneous Solution [Transient vibration]

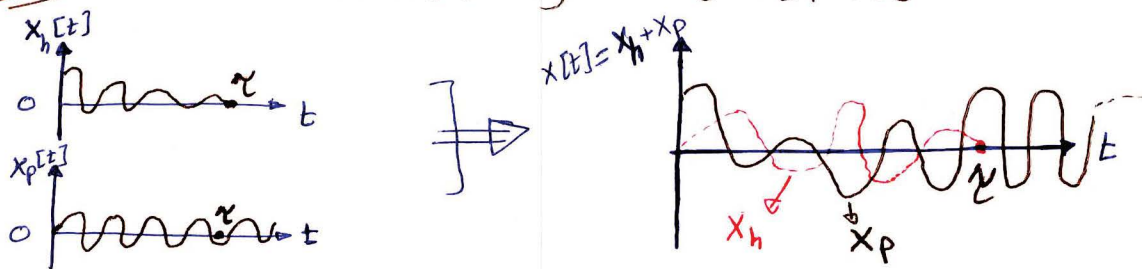
x_p is Particular Solution [Steady state vibration]

① x_h $m \ddot{x} + c \dot{x} + kx = 0$

$x_h = 000$ as in ch2

- 3 cases - underdamped
- critical damping
- overdamping

② x_p $\leadsto$ represents the steady-state response



* x_h dies out and $x[t]$ becomes x_p after sometime (ψ)

* $k, c,$ and m $\leftarrow$ system parameters يعقد على قيم ال Transient motion



3.3 Response of an Undamped System Under Harmonic Force

Undamped Forced [Harmonic]

$$\boxed{K} m\ddot{x} + Kx = F_0 \cos \omega t$$

remember $\rightarrow r = \frac{\omega}{\omega_n}$

General Solution $\rightarrow X[t] = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \underline{X} \cos \omega t$

If $\begin{cases} r < 1 \\ r > 1 \end{cases}$

where $\underline{X} = \frac{\delta_{st}}{1-r^2} = \frac{F_0}{k - m\omega^2}$

represents the ratio
of the dynamic to the
static amplitude
of motion

$\frac{\underline{X}}{\delta_{st}} = M_3$ magnification factor
or amplification factor
or amplitude ratio

$$C_1 = X_0 - \frac{F_0}{k - m\omega^2} \quad , \quad C_2 = \frac{\dot{X}_0}{\omega_n}$$

$$F_0 = \delta_{st} K$$

General
Solution

$$\rightarrow X[t] = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t$$

If $(r=1)$

where $C_1 = X_0 \quad , \quad C_2 = \frac{\dot{X}_0}{\omega_n}$



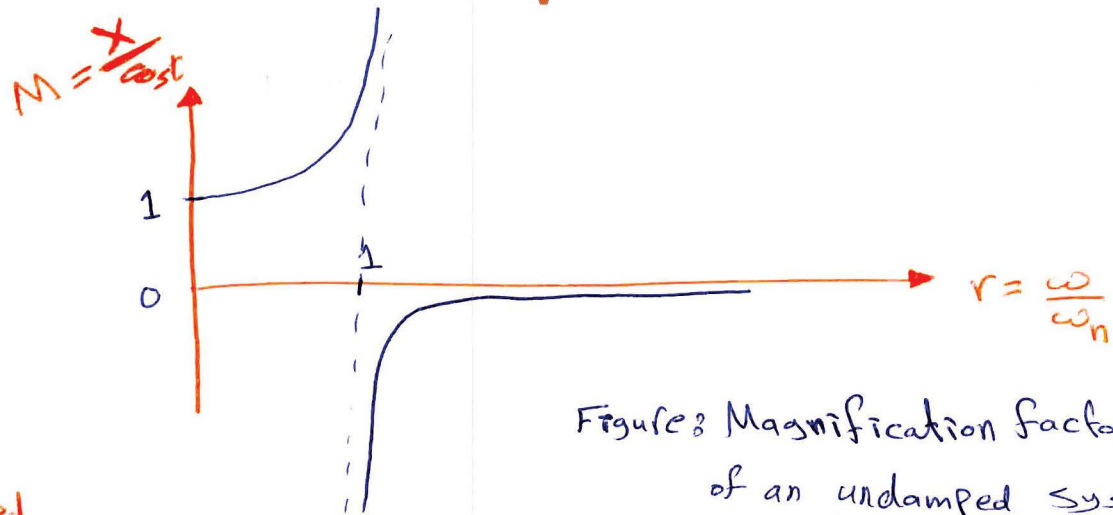


Figure: Magnification factor of an undamped system

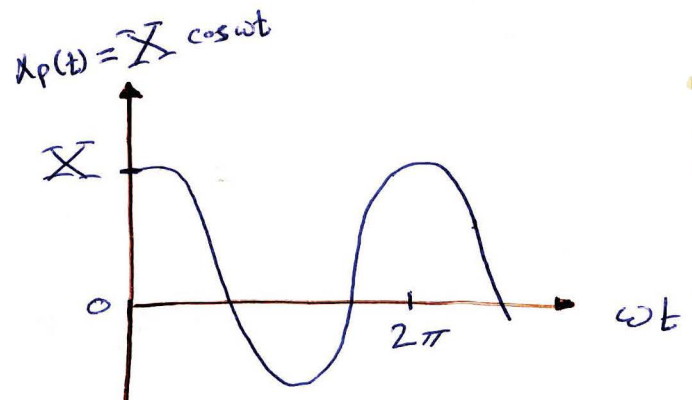
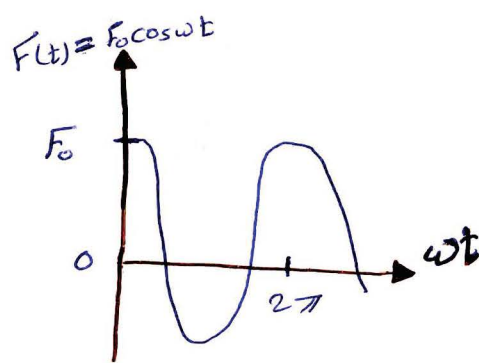
Undamped

Case 1 $0 < r < 1$

$$\frac{X}{\delta_{st}} = \frac{1}{1 - r^2} \rightarrow \text{denominator (المقام)}$$

⊛ denominator is +ve

⊛ The response is given by $\Rightarrow x_p(t) = X \cos \omega t$ without change



Case 2 $r > 1$

$$\frac{X}{\delta_{st}} = \frac{1}{1-r^2}$$

⊛ denominator is -ve

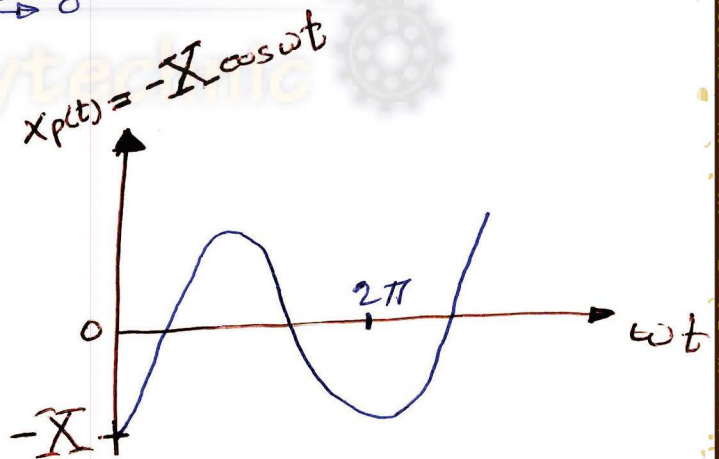
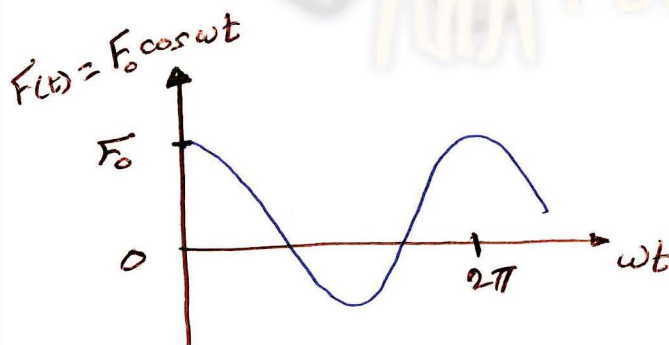
⊛ The steady-state solution can be expressed as $\Rightarrow x_p(t) = -X \cos \omega t$

where X is redefined to be a positive quantity $\Rightarrow X = \frac{\delta_{st}}{r^2-1}$

⊛ $x_p(t)$ & $F(t)$ have opposite signs,

so $\rightarrow$ The response is said to be 180° out of phase with the external force

$r = \frac{\omega}{\omega_n} \rightarrow \infty$, $X \rightarrow 0$



Case 3 $r=1$ [at resonance]

$$(*) \quad \bar{X} = \frac{\delta_{st}}{1-r^2} \text{ or } X = \frac{\delta_{st}}{r^2-1}$$

$$r=1 \rightarrow X = \infty$$

(*) The response for this condition

is

$$X(t) = X_0 \cos \omega_n t + \frac{\dot{X}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t$$

(*) لمعرفة كيف تم الحصول على هذا القانون

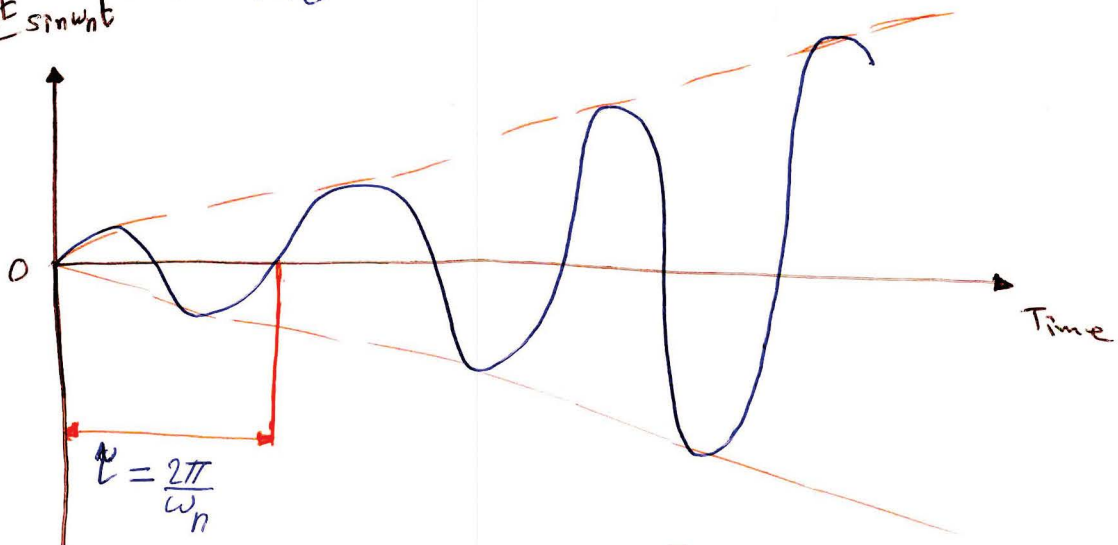
ارجع الالاميات صفحة 265 & 266 من الكتاب المعتقد

- يلاحظ أنه تم استخدام قاعدة L'Hospital's rule في النهاية لأن البسط = 0
Zero = 0 / Zero = 0

(*) The last term of ^{above} equation $\left[\frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \right]$
amplitude

→ The amplitude can be seen to increase linearly

$$x_p = \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \text{ with Time}$$



3.3.2. Beating phenomenon

* Beating phenomenon takes place

when ω is close to [not exactly equal to] ω_n

* The amplitude builds up & then diminishes in regular pattern [see Section 1.10.5 "page 63"]

* If initial conditions are taken as

$$x_0 = \dot{x}_0 = 0$$

The equation $\rightarrow x(t) = \left[x_0 - \frac{F_0}{k - m\omega_n^2} \right] \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$

reduces to

$$\rightarrow x(t) = \frac{F_0/m}{\omega_n^2 - \omega^2} [\cos \omega t - \cos \omega_n t]$$

$$x(t) = \frac{F_0/m}{\omega_n^2 - \omega^2} \left[2 \sin \frac{\omega + \omega_n}{2} t * \sin \frac{\omega_n - \omega}{2} t \right]$$

Let ω be slightly less than ω_n

$$\omega_n - \omega = 2\varepsilon$$

$$\omega + \omega_n \approx 2\omega$$

$$\omega_n^2 - \omega^2 = 4\varepsilon\omega$$

عوضهم في

يصبح

$$x(t) = \left(\frac{F_0/m}{2\varepsilon\omega} \sin \varepsilon t \right) \sin \omega t$$

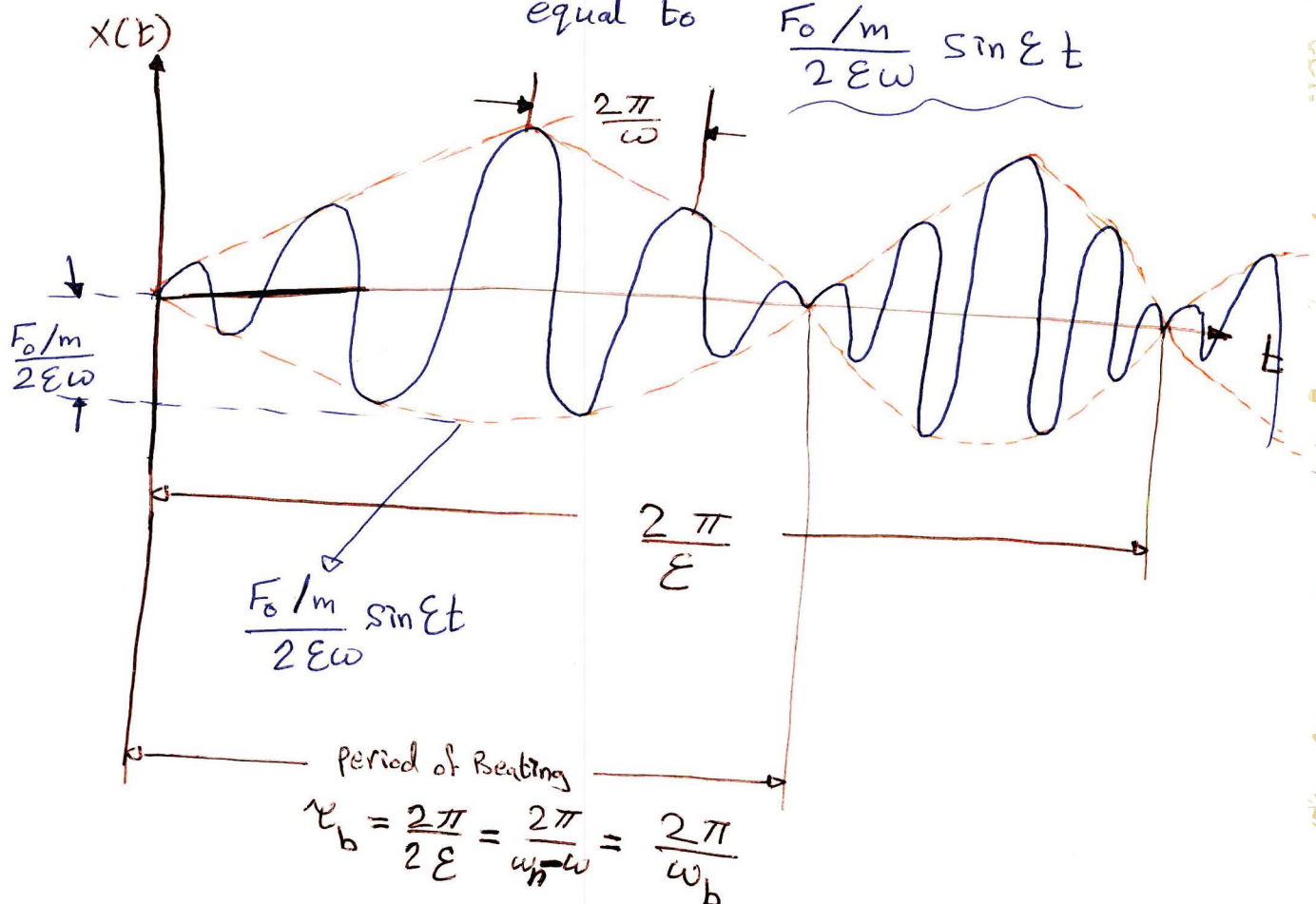
where ε : small positive quantity



$$x(t) = \left[\frac{F_0/m}{2\varepsilon\omega} \sin \varepsilon t \right] \sin \omega t$$

- Since ε is small, The function $\sin \varepsilon t$ varies slowly; its period, equal to $\boxed{\frac{2\pi}{\varepsilon}}$, is large.

Thus Above equation $\rightarrow$ may be seen as representing vibration with period $\boxed{\frac{2\pi}{\omega}}$ and of variable amplitude equal to $\frac{F_0/m}{2\varepsilon\omega} \sin \varepsilon t$



⊛ where ω_b - frequency of beating



again

$$X(t) = \left(\frac{F_0/m}{2\varepsilon\omega} \sin \varepsilon t \right) \sin \omega t$$

- ⊗ It can be observed that $\sin \omega t$ curve will go through several cycles, while the $\sin \varepsilon t$ wave goes through a single cycles
- ⊗ Amplitude builds up and dies down continuously.
- ⊗ The time between the points of zero amplitude or the points of maximum amplitude is called **The period of beating τ_b**

$$\rightarrow \tau_b = \frac{2\pi}{2\varepsilon} = \frac{2\pi}{\omega_b - \omega}$$

$$\omega_b = 2\varepsilon = \omega_n - \omega$$



3.4 Response of a Damped System Under Harmonic Force

⊠ Forced "Harmonic", Damped

equation

$$\rightarrow m \ddot{x} + c \dot{x} + kx = F_0 \cos \omega t$$

Solution
For Undamped $\rightarrow x(t) = \overset{-2\omega_n t}{X_0} e^{\dots} \cos[\omega_d t - \phi] + \underline{X} \cos[\omega t - \phi]$

where

$$\underline{X} = \frac{\delta_{sb}}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} = \frac{F_0}{\sqrt{[k-m\omega^2]^2 + c^2\omega^2}}$$

$$\phi = \tan^{-1} \left[\frac{2\zeta r}{1-r^2} \right] = \tan^{-1} \left[\frac{c\omega}{k-m\omega^2} \right]$$

$$\underline{X}_0 = \sqrt{[x_0 - \underline{X} \cos \phi]^2 + \frac{1}{\omega_d^2} [\zeta \omega_n x_0 + \dot{x}_0 - \zeta \omega_n \underline{X} \cos \phi - \omega \underline{X} \sin \phi]^2}$$

$$\phi_0 = \frac{\tan^{-1} [\zeta \omega_n x_0 + \dot{x}_0 - \zeta \omega_n \underline{X} \cos \phi - \omega \underline{X} \sin \phi]}{\omega_d [x_0 - \underline{X} \cos \phi]}$$



☒ look at Figure 3.10 a and b in page 272

☒ look at Figure 3.11 a and b in page 273
and read The Characteristics

3.4.2 Quality factor and Bandwidth

$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



☒ For small values of damping [$\zeta < 0.05$]
we can take

$$\left[\frac{X}{\delta_{st}} \right]_{\max} \approx \left[\frac{X}{\delta_{st}} \right]_{\omega=\omega_n} = \frac{1}{2\zeta} = Q$$

☒ The value of amplitude ratio at resonance [$r=1$]
is also called the Q Factor or quality factor
of the system



⊠ The points R_1 & R_2 , where the amplification factor falls to $\frac{Q}{\sqrt{2}}$ are called **Half-power points**

- Because the power absorbed (ΔW) by the damper, responding harmonically at a given frequency, is proportional to the square of the amplitude

$$\rightarrow \Delta W = \pi c \omega X^2$$

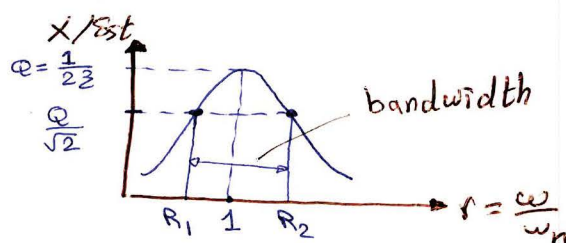


Figure 2 Half-power points

⊠ The difference between the frequencies associated with the half-power points R_1 & R_2 is called **The Bandwidth**

⊠ To Find R_1 & R_2 , we set $\frac{X}{X_{st}} = \frac{Q}{\sqrt{2}}$,

in equation $\rightarrow \frac{X}{X_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$

or $r^4 - r^2[2 - 4\zeta^2] + [1 - 8\zeta^2] = 0$

The solution

$\rightarrow r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1+\zeta^2}$, $r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1+\zeta^2}$

⊠ For small values of ζ

$$r_1^2 = R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1 - 2\zeta^2$$

$$r_2^2 = R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1 + 2\zeta^2$$



where $\omega_1 = \omega / R_1$, $\omega_2 = \omega / R_2$

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$$\omega_2^2 - \omega_1^2 = (\omega_2 + \omega_1)(\omega_2 - \omega_1)$$

$$= (R_2^2 - R_1^2) \omega_n^2 \approx 4\zeta \omega_n^2$$

Using $(\omega_2 + \omega_1 = 2\omega_n)$ in $\rightarrow$

we find Bandwidth

$$\Delta\omega = \omega_2 - \omega_1 \approx 2\zeta \omega_n$$

بمع الترتيب $\rightarrow$

$$\frac{1}{2\zeta} = \frac{\omega_n}{\omega_2 - \omega_1} \quad \text{--- (1)}$$

if $\zeta < 0.05$
Remember

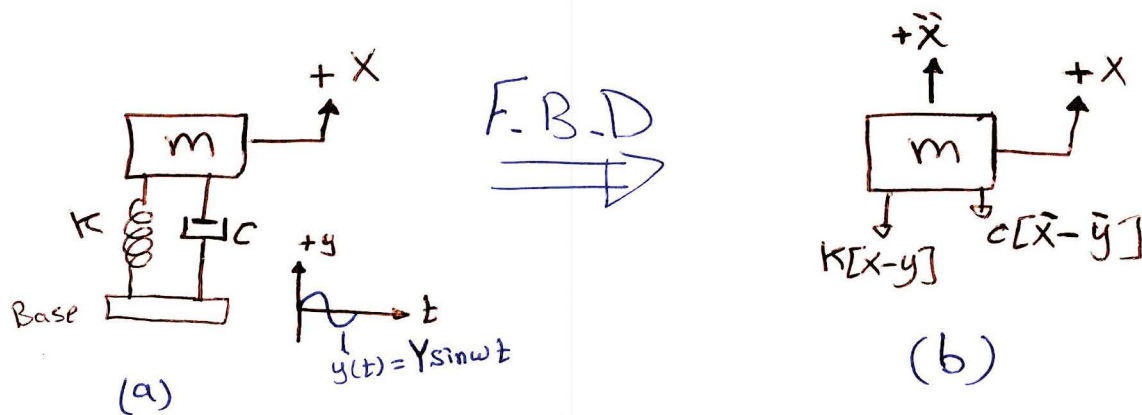
$$Q = \frac{1}{2\zeta} \quad \text{--- (2)}$$

لو د عو معادلة 1 و 2

$$Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\omega_2 - \omega_1}$$

3-6 Response of a Damped System Under the Harmonic Motion of the Base

✱ Sometimes the base or support of a spring-mass-damper system undergoes harmonic motion as shown.



Figures Base Excitation

✱ Let $y(t)$ denotes the displacement of the base
 $x(t)$ denotes the displacement of the mass

From its static equilibrium position at time t

✱ Equation of Motion

$$m\ddot{x} + c[\dot{x}-\dot{y}] + k[x-y] = 0, \text{ If } \underline{y(t) = Y \sin \omega t}$$

$$m \ddot{x} + c \dot{x} + kx = ky + c\dot{y}$$

$$= kY \sin \omega t + c\omega Y \cos \omega t$$

$$= A \sin [\omega t - \alpha]$$

$$A = Y \sqrt{k^2 + (c\omega)^2}$$

$$\alpha = \tan^{-1} \left[\frac{c\omega}{k} \right]$$



where

قليلة

الاستقرار

$$x_p = X \sin [\omega t - \phi]$$

where

$$T_d = \frac{X}{Y} = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{1 + [2\zeta r]^2}{[1 - r^2]^2 + [2\zeta r]^2}$$

(displacement transmissibility)

$$\phi = \tan^{-1} \frac{m c \omega^3}{k[k - m\omega^2] + (c\omega)^2} = \tan^{-1} \frac{2\zeta r^3}{1 + [4\zeta^2 - 1]r^2}$$

is the ratio of the amplitude of the response $x_p(t)$ to that of the base motion $y(t)$

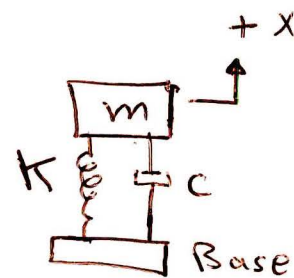


⊠ Look at Figure 3.15 a and b Page 283

and Read the aspects followed

3.6.1 Force Transmitted

⊠ In Figure, a Force, F , is transmitted to the base or support due to the reactions from the spring and the dashpot [damper]



This Force can be determined as

$$\rightarrow F = K[x-y] + c[\dot{x}-\dot{y}] = -m\ddot{x} \quad \text{---} (*)$$

From equation

$$x_p(t) = X \sin[\omega t - \phi]$$

The equation (*) can be written as

$$F = m\omega^2 X \sin[\omega t - \phi]$$

$$F = F_T \sin[\omega t - \phi]$$

where

F_T is the amplitude / maximum value of the force transmitted to the base

given by

(The Force Transmissibility) $\left(\frac{F_T}{KY} \right) = r^2 \sqrt{\frac{1 + [2\zeta r]^2}{[1-r^2]^2 + [2\zeta r]^2}}$



[*] Note that the transmitted force is in phase with the motion of the mass $X(t)$

[*] Look at Figure 3.16 page 284

3.6.2 Relative Motion

⊗ If $Z = X - y$ denotes the motion of mass relative to base

[*] Equation of Motion

$$m\ddot{x} + c[\dot{x} - \dot{y}] + k[x - y] = 0 \quad -m\ddot{y} \quad \left(\begin{array}{l} \text{قوة كلا} \\ \text{الطرفين} \end{array} \right)$$

It becomes $\rightarrow m\ddot{Z} + c\dot{Z} + kZ = -m\ddot{y}$

$$\rightarrow m\ddot{Z} + c\dot{Z} + kZ = m\omega^2 Y \sin\omega t$$

- The steady state of $\uparrow$

$$\rightarrow Z(t) = Z \sin[\omega t - \phi_1]$$

where Z is the amplitude of $Z(t)$

$$\left\{ \begin{array}{l} Z = Y = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \\ \phi_1 = \tan^{-1} \left[\frac{2\zeta r}{1-r^2} \right] \end{array} \right.$$

[*] Look at Figure 3.17 and ϕ_1 diagram like ϕ in Figure 3.11 b



3.7 Response of a Damped System Under Rotating Unbalance

[*] Unbalance in rotating machinery is one of the main causes of vibration

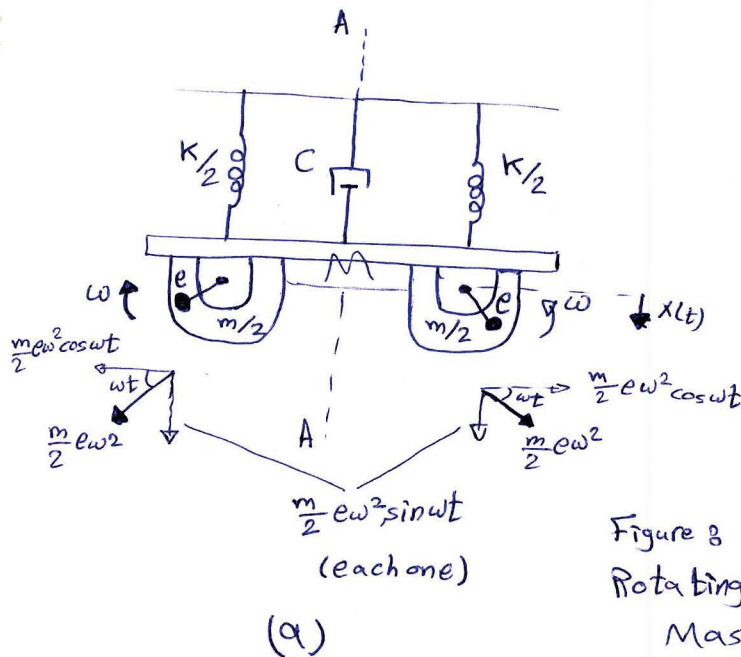
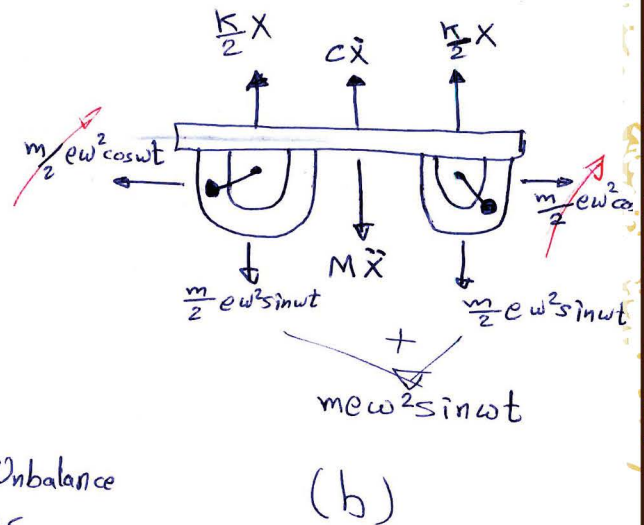


Figure 3
Rotating Unbalance
Masses



[*] M is total mass of the machine

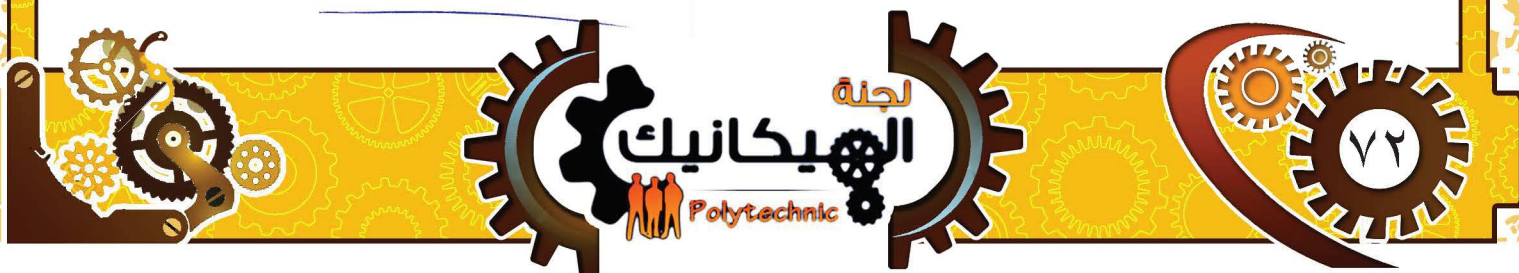
$\frac{m}{2}$ is two eccentric masses rotating in opposite directions with constant angular velocity ω

$\frac{m}{2} e \omega^2$ is centrifugal force due to each mass will cause excitation of the mass

Pr. $\frac{F_c}{m} = a_c = \frac{v^2}{r}$ $v = \omega r$

$$F_c = m a_c = \frac{m}{2} * \frac{v^2}{e} = \frac{m}{2} * \frac{\omega^2 * e^2}{e}$$

$$F_c = \frac{m}{2} e \omega^2 \rightarrow \text{centrifugal force}$$



Under Rotating Unbalance Damped

* The equation $\rightarrow M\ddot{X} + C\dot{X} + KX = me\omega^2 \sin \omega t$

$$\rightarrow X_p = \bar{X} \sin[\omega t - \phi]$$

where

$$\bar{X} = \frac{me\omega^2}{\sqrt{[K - M\omega^2]^2 + [C\omega]^2}} \quad \text{(OR)} \quad \frac{M\bar{X}}{me} = \frac{r^2}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}}$$

$$\phi = \tan^{-1} \frac{C\omega}{K - M\omega^2} \quad \left(\phi = \tan^{-1} \frac{2\zeta r}{1-r^2} \right)$$

* Maximum $\frac{M\bar{X}}{me}$ occurs only when $0 < \zeta < \frac{1}{\sqrt{2}}$

$$\text{at } r = \frac{1}{\sqrt{1-2\zeta^2}} > 1$$

$$\left[\frac{M\bar{X}}{me} \right]_{\max} = \frac{1}{2\zeta \sqrt{1-\zeta^2}}$$



⊗ Force Transmitted to the Foundation
due to rotating unbalance Force "F"

Can be found as

$$\hookrightarrow F(t) = KX(t) + C\dot{X}(t)$$

⊗ Magnitude / maximum value

of $\underline{F}$ $\hookrightarrow |F| = m\epsilon\omega^2 \sqrt{\frac{1 + 4\zeta^2 r^2}{[1 - r^2]^2 + 4\zeta^2 r^2}}$

⊗ Note

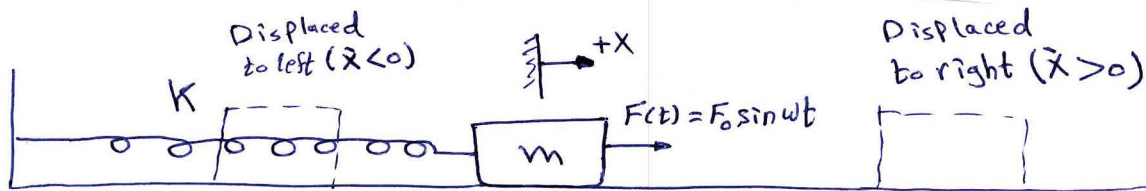
velocity = wavelength * frequency

$$\boxed{v = \lambda * f}$$

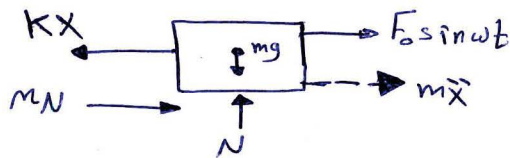


3.8 Forced Vibration with Coulomb Damping

* The equation $\rightarrow m\ddot{x} + Kx \pm MN = F_0 \sin \omega t$



(a)



(b)



$$x_p = X \sin[\omega t - \phi]$$

where
$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c_{eq}\omega)^2}} = \frac{(F_0/K)}{\sqrt{(1-r^2)^2 + (2\zeta_{eq}r)^2}}$$

or
$$X = \frac{F_0}{K} \sqrt{\frac{1 - \left[\frac{4MN}{\pi F_0}\right]^2}{(1-r^2)^2}}$$

$$\phi = \tan^{-1} \left[\frac{c_{eq}\omega}{k - m\omega^2} \right] = \tan^{-1} \left[\frac{2\zeta_{eq}r}{1-r^2} \right] = \tan^{-1} \left[\frac{\frac{4MN}{\pi K X}}{1-r^2} \right]$$

or
$$\phi = \tan^{-1} \left[\frac{\frac{4MN}{\pi F_0}}{\sqrt{1 - \left[\frac{4MN}{\pi F_0}\right]^2}} \right]$$

→



where

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$$z_{eq} = \frac{C_{eq}}{C_{cr}} = \frac{C_{eq}}{2m\omega_n} = \frac{4MN}{2m\omega_n \pi \omega X} = \frac{2MN}{\pi m \omega \omega_n X}$$

ϕ is discontinuous at $r=1$

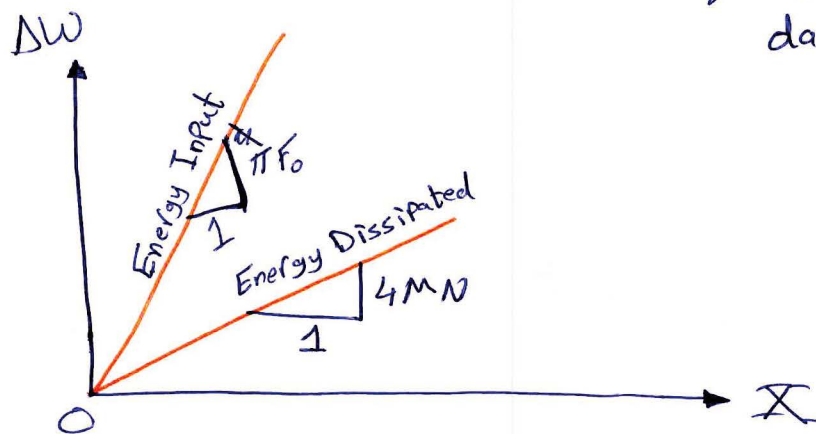
ϕ is positive [+ve] when $r < 1$

ϕ is negative [-ve] when $r > 1$

$$\rightarrow C_{eq} = \frac{4MN}{\pi \omega X}$$

where

C_{eq} : equivalent viscous damping constant



⊛ Energy Input to System per cycle $\Rightarrow \boxed{\Delta W' = \pi F_0 X}$

⊛ Energy Dissipated Per cycle $\Rightarrow \boxed{\begin{aligned} \Delta W &= 4MN X \\ \Delta W &= \pi C_{eq} \omega X \end{aligned}}$

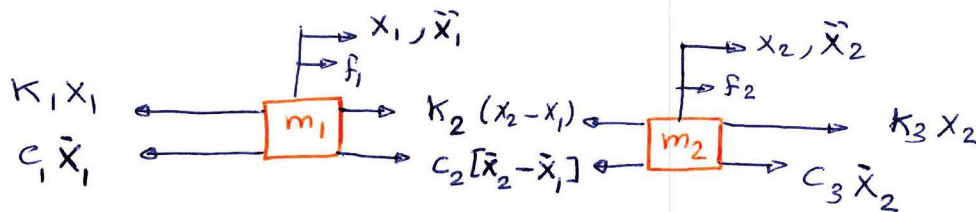
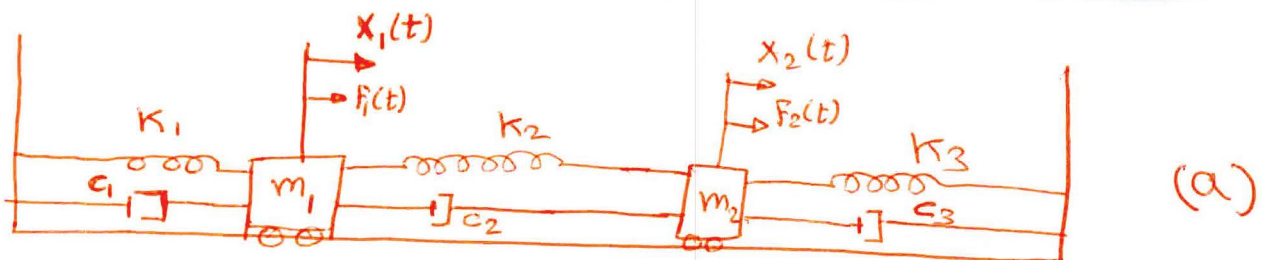


CH5 Two - Degree - of - Freedom Systems

5.1 Introduction عليك قراءة

from page 468 ~ 472

5.2 Equations of Motion for forced Vibration



Spring k_1 under tension for $+x_1$

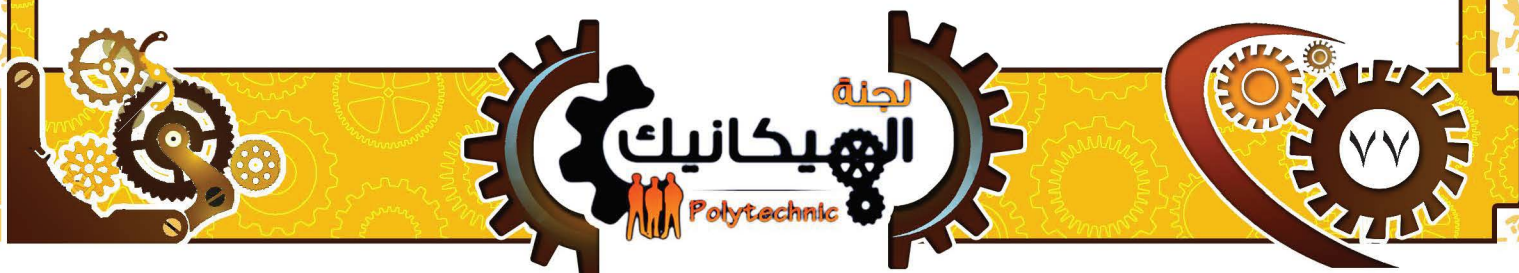
Spring k_2 under tension for $+(x_2 - x_1)$

Spring k_3 under compression for $+x_2$

Figure 8 A two-degree-of-freedom spring-mass-damper system

مراجعة

Page 87, 88 القوانين العامة



* Equations

$$① \rightarrow m_1 \ddot{x}_1 + [c_1 + c_2] \dot{x}_1 - c_2 \dot{x}_2 + [k_1 + k_2] x_1 - k_2 x_2 = f_1$$

$$② \rightarrow m_2 \ddot{x}_2 - c_2 \dot{x}_1 + [c_2 + c_3] \dot{x}_2 - k_2 x_1 + [k_2 + k_3] x_2 = f_2$$

matrix

$$[m] \ddot{\vec{x}} + [C] \dot{\vec{x}} + [K] \vec{x} = \vec{f}(t)$$

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad [C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \rightarrow \text{Displacement vector}$$

$$\vec{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \rightarrow \text{Force vector}$$

* $[m], [C], [K] \rightarrow$ are all 2×2 matrices
 matrix 2x2

* Further, these matrices can be seen to be Symmetric

where

T : Transpose

$$[m]^T = [m], [C]^T = [C]$$

$$[K]^T = [K]$$

يفضل في في مادة ال

وإن لم يكن مفهوم فإن يُفهر



⊠ The equations (1) & (2)

→ Become uncoupled [independent of one another]
only when $C_2 = k_2 = 0$

which implies that the two masses are not physically connected
يعني

⊠ The Solution involves 4 constants of integration

⊠ Usually the initial displacements & velocities of the two masses are specified as -
 $x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)$

5.3 Free-Vibration Analysis of an Undamped System

For Free → $F_1(t) = F_2(t) = 0$

without damping → $C_1 = C_2 = C_3 = 0$

no. of eqn.
in Book
4

(1) & (2) become

$$\rightarrow m_1 \ddot{x}_1(t) + [k_1 + k_2] x_1(t) - k_2 x_2(t) = 0 \quad \dots \underline{5.4}$$

$$\rightarrow m_2 \ddot{x}_2(t) - k_2 x_1(t) + [k_2 + k_3] x_2(t) = 0 \quad \dots \underline{5.5}$$

⊠ Assume m_1 & m_2 oscillate harmonically with the same ω & ϕ , Then the solutions

$$\rightarrow x_1(t) = X_1 \cos[\omega t + \phi]$$

$$x_2(t) = X_2 \cos[\omega t + \phi]$$

} ... 5.6



* Substitute 5.6 into 5.4 & 5.5

$$\rightarrow \left[\left[-m_1 \omega^2 + [k_1 + k_2] \right] X_1 - k_2 X_2 \right] \cos[\omega t + \phi] = 0$$

$$\left[-k_2 X_1 + \left[-m_2 \omega^2 + [k_2 + k_3] \right] X_2 \right] \cos[\omega t + \phi] = 0$$

$$\rightarrow \left. \begin{aligned} \left[-m_1 \omega^2 + [k_1 + k_2] \right] X_1 - k_2 X_2 &= 0 \\ -k_2 X_1 + \left[-m_2 \omega^2 + [k_2 + k_3] \right] X_2 &= 0 \end{aligned} \right\} \underline{\underline{5.8}}$$

- Trivial solution $\rightarrow X_1 = X_2 = 0$
 لا تتأق / سغيف ٠٠٠

- nontrivial solution $\rightarrow$ det. of X_1 & $X_2 = 0$
 مائو

OR

$$\rightarrow [m_1 m_2] \omega^4 - [[k_1 + k_2] m_2 + [k_2 + k_3] m_1] \omega^2 + [[k_1 + k_2] [k_2 + k_3] - k_2^2] = 0 \quad \underline{\underline{5.9}}$$

* Equation 5.9 is called $\rightarrow$ The Frequency
or Characteristic equation



* The roots of equation 5.9

[81]

$$\omega_1^2, \omega_2^2 = \frac{1}{2} \left[\frac{[k_1 + k_2]m_2 + [k_2 + k_3]m_1}{m_1 m_2} \right] \pm \frac{1}{2} \left[\frac{[k_1 + k_2]m_2 + [k_2 + k_3]m_1}{m_1 m_2} \right]^2 - 4 \left[\frac{[k_1 + k_2][k_2 + k_3] - k_2^2}{m_1 m_2} \right]^{1/2}$$

000 5.10

* ω_1 & ω_2 are called natural frequencies of the system

ملاحظة ...
القيمة الأقل تكون ω_1
والأعلى تكون ω_2

* The values of X_1 & X_2

~ corresponding to ω_1 as $\rightarrow X_1^{(1)}$ & $X_2^{(1)}$

~ corresponding to ω_2 as $\rightarrow X_1^{(2)}$ & $X_2^{(2)}$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} \quad , \quad r_2 = \frac{X_2^{(2)}}{X_1^{(2)}}$$



* For $\omega^2 = \omega_1^2$ & $\omega^2 = \omega_2^2$

equation 5.8

gives $\rightarrow$ $r_1 = \frac{X_2^1}{X_1^1} = \frac{-m_1 \omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + [k_2 + k_3]}$

$\rightarrow$ $r_2 = \frac{X_2^2}{X_1^2} = \frac{-m_1 \omega_2^2 + [k_1 + k_2]}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + [k_2 + k_3]}$

* The normal modes of vibration corresponding to ω_1^2 & ω_2^2

$$\rightarrow \vec{X}^{(1)} = \begin{bmatrix} X_1^1 \\ X_2^1 \end{bmatrix} = \begin{bmatrix} X_1^1 \\ r_1 X_1^1 \end{bmatrix}$$

$$\vec{X}^{(2)} = \begin{bmatrix} X_1^2 \\ X_2^2 \end{bmatrix} = \begin{bmatrix} X_1^2 \\ r_2 X_1^2 \end{bmatrix}$$

also known as modal vectors of system



✱ The Free-Vibration Solution of the motion in time
Can be expressed, using equation 5.6

as

$$\vec{X}^{(1)}(t) = \begin{bmatrix} X_1^1(t) \\ X_2^1(t) \end{bmatrix} = \begin{bmatrix} X_1^1 \cos[\omega_1 t + \phi_1] \\ r_1 X_1^1 \cos[\omega_1 t + \phi_1] \end{bmatrix} = \text{First mode}$$

$$\vec{X}^{(2)}(t) = \begin{bmatrix} X_1^2(t) \\ X_2^2(t) \end{bmatrix} = \begin{bmatrix} X_1^2 \cos[\omega_2 t + \phi_2] \\ r_2 X_1^2 \cos[\omega_2 t + \phi_2] \end{bmatrix} = \text{Second mode}$$

✱ where $X_1^1, X_1^2, \phi_1, \phi_2$ are determined by the initial condition

$$x_1(0) = X_1^{(i)} = \text{Some Constant}, \quad \dot{x}_1(0) = 0$$

$$x_2(0) = r_1 X_1^{(i)}, \quad \dot{x}_2(0) = 0$$

✱ The general Solution of equation 5.4 & 5.5

Can be obtained by a linear superposition
of the two normal modes, equation 5.13

$$\vec{X}(t) = C_1 \vec{X}^{(1)} + C_2 \vec{X}^{(2)}(t)$$

we can choose $C_1 = C_2 = 1$ (with no loss
of generality)



Thus ^{بالتالي}

$$\rightarrow X_1(t) = X_1^1(t) + X_1^2(t) = X_1^1 \cos[\omega_1 t + \phi_1] + X_1^2 \cos[\omega_2 t + \phi_2]$$

$$\rightarrow X_2(t) = X_2^1(t) + X_2^2(t) = r_1 X_1^1 \cos[\omega_1 t + \phi_1] + r_2 X_1^2 \cos[\omega_2 t + \phi_2]$$

ooo equ. 5.15

where $X_1^1, X_1^2, \phi_1, \phi_2$ ooo from initial conditions

$$X_1(t=0) = X_1(0), \quad \dot{X}_1(t=0) = \dot{X}_1(0)$$

$$X_2(t=0) = X_2(0), \quad \dot{X}_2(t=0) = \dot{X}_2(0)$$

ooo 5.16

⊠ Substitute 5.16 into 5.15

$$X_1^{(1)} \cos \phi_1 = \left[\frac{r_2 X_1(0) - X_2(0)}{r_2 - r_1} \right], \quad X_1^{(2)} \cos \phi_2 = \left[\frac{-r_1 X_1(0) + X_2(0)}{r_2 - r_1} \right]$$

$$X_1^{(1)} \sin \phi_1 = \left[\frac{-r_2 \dot{X}_1(0) + \dot{X}_2(0)}{\omega_1 [r_2 - r_1]} \right], \quad X_1^{(2)} \sin \phi_2 = \left[\frac{r_1 \dot{X}_1(0) - \dot{X}_2(0)}{\omega_2 [r_2 - r_1]} \right]$$

$$X_1^{(1)} = \sqrt{[X_1^{(1)} \cos \phi_1]^2 + [X_1^{(1)} \sin \phi_1]^2}$$

$$X_1^{(2)} = \sqrt{[X_1^{(2)} \cos \phi_2]^2 + [X_1^{(2)} \sin \phi_2]^2}$$

$$\phi_1 = \tan^{-1} \frac{X_1^{(1)} \sin \phi_1}{X_1^{(1)} \cos \phi_1}, \quad \phi_2 = \tan^{-1} \frac{X_1^{(2)} \sin \phi_2}{X_1^{(2)} \cos \phi_2}$$



5.4 Torsional System

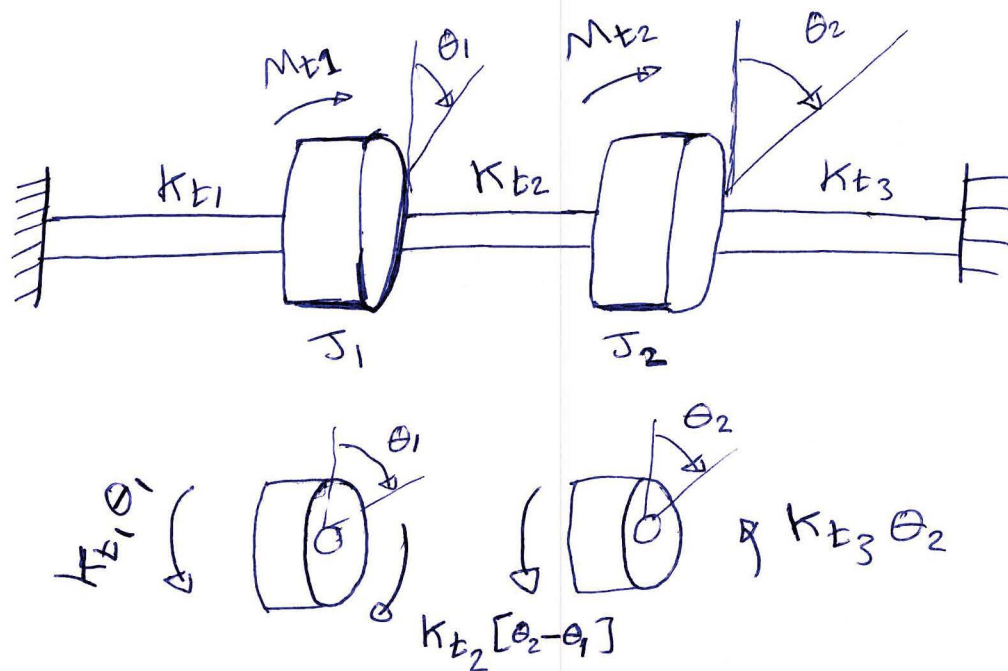


Figure 3 Torsional system with discs mounted on a shaft

[*] The Diff. Equ. of rotational motion

for discs J_1 & J_2 can be derived as:-

$$J_1 \ddot{\theta}_1 = -k_{t1} \theta_1 + k_{t2} [\theta_2 - \theta_1] + M_{t1}$$

$$J_2 \ddot{\theta}_2 = -k_{t2} [\theta_2 - \theta_1] - k_{t3} \theta_2 + M_{t2}$$



⊠ which upon rearrangement become

$$\rightarrow J_1 \ddot{\theta}_1 + [k_{t1} + k_{t2}] \theta_1 - k_{t2} \theta_2 = M_{t1}$$

$$J_2 \ddot{\theta}_2 - k_{t2} \theta_1 + [k_{t2} + k_{t3}] \theta_2 = M_{t2}$$

⊠ For Free ... $M_{t1} = M_{t2} = 0$

$$\rightarrow J_1 \ddot{\theta}_1 + [k_{t1} + k_{t2}] \theta_1 - k_{t2} \theta_2 = 0$$

$$\rightarrow J_2 \ddot{\theta}_2 - k_{t2} \theta_1 + [k_{t2} + k_{t3}] \theta_2 = 0$$

5.20

⊠ 5.20 is similar to equ. 5.4 & 5.5

⊠ In fact ...

equ. 5.20 can be obtained by substitution

$\theta_1, \theta_2, J_1, J_2, k_{t1}, k_{t2}$ and k_{t3}
for

$x_1, x_2, m_1, m_2, k_1, k_2$ and k_3

respectively.

على التوالي



CHAPTER 5 أنظمة المعادلات في

$$m_1 \ddot{x}_1 + [k_1 + k_2] x_1 - k_2 x_2 = 0$$

$$m_2 \ddot{x}_2 - k_2 x_1 + [k_2 + k_3] x_2 = 0$$

The Solution

$$x_1 = X_1 \cos[\omega t + \phi]$$

$$x_2 = X_2 \cos[\omega t + \phi]$$

To Find ω_1, ω_2

من المعادلات في الأعلى

$$\begin{bmatrix} -m_1 \omega^2 + [k_1 + k_2] & -k_2 \\ -k_2 & -m_2 \omega^2 + [k_2 + k_3] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{if } \omega^2 = \omega_1^2 \rightarrow x_2^1 = \frac{-m_1 \omega_1^2 + [k_1 + k_2]}{k_2} x_1^1$$

$$\text{if } \omega^2 = \omega_2^2 \rightarrow x_2^2 = \frac{-m_1 \omega_2^2 + [k_1 + k_2]}{k_2} x_1^2$$

$$r_1 = \frac{x_2^1}{x_1^1}$$

$$r_2 = \frac{x_2^2}{x_1^2}$$



First mode $\rightarrow \vec{X}^1(t) = \begin{bmatrix} X_1^1(t) \\ X_2^1(t) \end{bmatrix} = \begin{bmatrix} X_1^1 \cos[\omega_1 t + \phi_1] \\ r_1 X_1^1 \cos[\omega_1 t + \phi_1] \end{bmatrix}$

Second mode $\rightarrow \vec{X}^2(t) = \begin{bmatrix} X_1^2(t) \\ X_2^2(t) \end{bmatrix} = \begin{bmatrix} X_1^2 \cos[\omega_2 t + \phi_2] \\ r_2 X_1^2 \cos[\omega_2 t + \phi_2] \end{bmatrix}$

* The General Solution

$$X_1(t) = X_1^1(t) + X_1^2(t) = X_1^1 \cos[\omega_1 t + \phi_1] + X_1^2 \cos[\omega_2 t + \phi_2]$$

$$X_2(t) = X_2^1(t) + X_2^2(t) = X_2^1 \cos[\omega_1 t + \phi_1] + X_2^2 \cos[\omega_2 t + \phi_2]$$

where

$$X_2^1 = r_1 X_1^1$$

$$X_2^2 = r_2 X_1^2$$

note we can find X_1^1 , X_1^2 , ϕ_1 , ϕ_2

from initial condition $[X_1(0), \dot{X}_1(0), X_2(0), \dot{X}_2(0)]$

✧ إن كان لديك ملاحظات أو تعديله على ما تمّ كتابته

فيرجى التواصل مع لجنة الميكانيك

فالعصمة لكتاب الله فقط

والحمد لله رب العالمين

